Ancient Solutions to Geometric Flows

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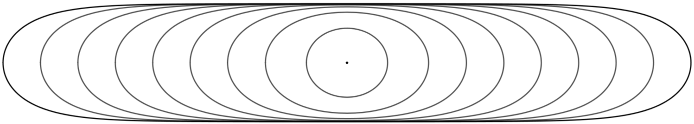
Geometric PDE Zoom Seminar

October 6, 2020.



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*Liouville's theorem actually holds assuming only a one-sided bound (which can be replaced by a sublinear growth rate).

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The result follows from the Cheng–Yau gradient estimate: If (M, g) satisfies $\mathrm{Rc} \geq -Kg$, $K > 0$, and $u \in C^\infty(B_r(x))$ is a positive harmonic function, then

$$\max_{B_{r/2}(x)} \frac{|\nabla u|}{u} \leq c_n \left(r^{-1} + \sqrt{K} \right).$$

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Similar methods were later used by Li–Yau and Hamilton to obtain differential Harnack inequalities for parabolic equations.

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There are counterexamples in dimensions 8 and higher [Simons, Bonbieri–de Giorgi–Giusti].

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$$\frac{|\nabla u(x,t)|}{u(x,t)} \leq c_n \left(r^{-1} + \sqrt{K} \right) \left(1 + \log \left(\frac{\sup_{P_r(x,t)} u}{u(x,t)} \right) \right) \quad \text{in} \quad P_{r/2}(x,t)$$

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The point here is that the **modulus of continuity**

$$\omega(s,t) := \sup \left\{ \frac{u(x,t)-u(y,t)}{2} : \frac{d(x,y)}{2} = s \right\}$$

of a solution to the heat equation is a subsolution to the one-dimensional heat equation (with induced boundary conditions).

Semilinear heat equations

Consider now solutions $u : \mathbb{R}^n \times (\alpha, \omega) \to \mathbb{R}$ to the **semi-linear heat equation**

$$\partial_t u = \Delta u + |u|^{p-1} u$$

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Theorem (Merle–Zaag) *Let* $u : \mathbb{R}^n \times (-\infty, \omega) \to \mathbb{R}$ *be a* **positive** *ancient solution to the semi-linear heat equation. If*

$$u(x,t) \leq O((\omega - t)^{-\frac{1}{p-1}}),$$

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Proof is based on the analysis of a Lyapunov functional.

Mean curvature flow

A family $\{\mathcal{M}_t^n\}_{t\in I}$ of hypersurfaces $\mathcal{M}_t^n \subset \mathbb{R}^{n+1}$ satisfies **mean curvature flow** if

$$\partial_t X(x,t) = \vec{H}(x,t)$$

for some parametrization $X : M^n \times I \to \mathbb{R}^{n+1}$, where $\vec{H}(\cdot,t) = \mathrm{div}(DX(\cdot,t))$ is the **mean curvature vector** of $\mathcal{M}_t^n$.

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If the time-slices $\mathcal{M}^n_t$ are mean convex boundaries, $\mathcal{M}^n_t = \partial\Omega_t$, then, equivalently, the **arrival time** $u : \cup_{t\in I}\mathcal{M}_t \to \mathbb{R}$ defined by

$$u(X) = t \iff X \in \mathcal{M}^n_t$$

satisfies the **level set flow**:

$$-|Du|\,\mathrm{div}\left(\frac{Du}{|Du|}\right) = 1\,.$$

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Indeed, suppose that $\lambda_j := |A_{(p_j,t_j)}| \to \infty$ as $j \to \infty$ for $p_j \in \mathcal{M}_{t_j}$, $t_j \in I$, and consider the rescaled flows

$$\mathcal{M}_t^j := \lambda_j(\mathcal{M}_{\lambda^{-2}t+t_j} - p_j)\,, \text{ for } t \in \lambda_j^2 I - t_j\,.$$

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See also [Bryan–Louie, Bryan–Ivaki–Scheuer, K. Choi–Mantoulidis, Huisken–Sinestrari, Lambert–Lotay–Schulze, L., L.–Lynch, Lynch–Nguyen, Risa–Sinestrari, Sonnanburg]

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The inequality is strict unless $\{\mathcal{M}_t^n\}_{t\in(0,\omega)}$ *is an* **expanding solution**:

$$\mathcal{M}_t=\sqrt{t}\mathcal{M}_1,\quad t>0.$$

Monotonicity formulae and self-similar solutions

Monotonicity of Gaussian area [Huisken]:

$$\frac{d}{dt}\int_{\mathcal{M}_t^n}(-4\pi t)^{-\frac{n}{2}}\mathrm{e}^{-\frac{|X|^2}{-4t}}\,dX \le -\int_{\mathcal{M}_t^n}\left|\vec{H}+\frac{X^\perp}{-2t}\right|^2(-4\pi t)^{-\frac{n}{2}}\mathrm{e}^{-\frac{|X|^2}{-4t}}\,dX\,.$$

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For ancient solutions,

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$$\mathcal{M}_t=\mathcal{M}_0+te,\quad t\in\mathbb{R}\ \text{ for some }\ e\in\mathbb{R}^{n+1}.$$

Shrinking solutions

For each pair of relatively prime (p, q) satisfying $\frac{1}{2} < \frac{p}{q} < \frac{\sqrt{2}}{2}$, there is a closed self-shrinking solution to curve shortening flow $\{\Gamma_t^{p,q}\}_{t\in(-\infty,0))}$ such that $\Gamma_{-1}^{p,q}$ has rotation index p, lies in an annulus about the origin, and touches each boundary of the annulus q times.

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Open question: Is Angenent's torus the only compact, embedded, genus 1 shrinker in $\mathbb{R}^3$?

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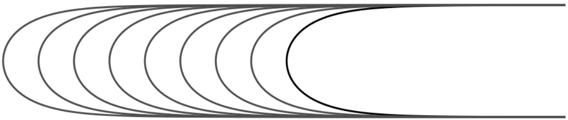

Figure: Snapshots of the Grim Reaper (with bulk velocity $v = e_1$).

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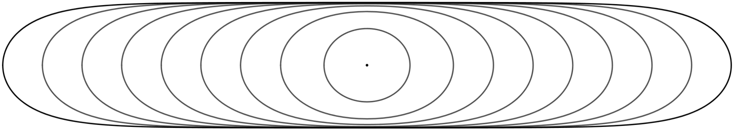

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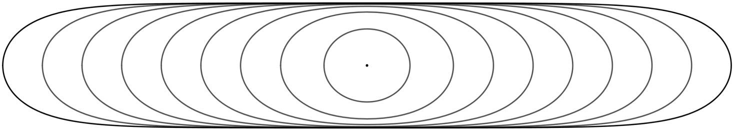

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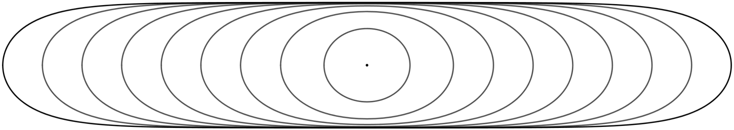

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N.b. Some authors require ancient ovaloids to be noncollapsing.

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- The Alexandrov reflection principle can then be used to show that the solution is either the Grim Reaper or the Angenent oval. □

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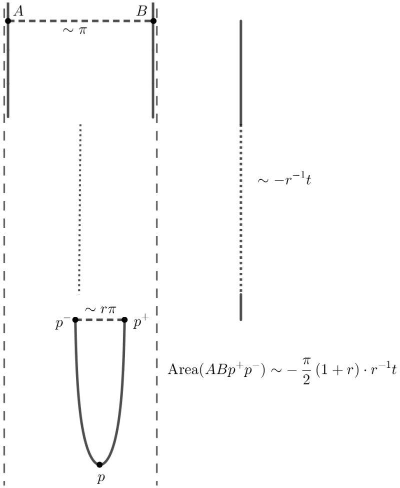

Figure: If the scale of the Grim Reaper forming at the tip is too small, its displacement, and hence also the enclosed area, is too large, since

$$\frac{d}{dt}\text{Area}(t) = -\int_A^B \kappa\, ds_t \quad \Longrightarrow \quad \text{Area}(t) \lesssim -\pi t.$$

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– *The Reapernoid* [Mramor–Payne]: explicit non-compact examples evolving out of catenoids (and certain other minimal hypersurfaces).

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Theorem (Angenent–Daskalopoulos–Šešum) $\{\mathcal{O}_t^{n,1}\}_{t \in (-\infty, 0)}$ *is the only ancient ovaloid in* $\mathbb{R}^{n+1}$ *which is* **noncollapsing** *and (when* $n \geq 3$*)* **uniformly two-convex**.

Ancient ovaloids

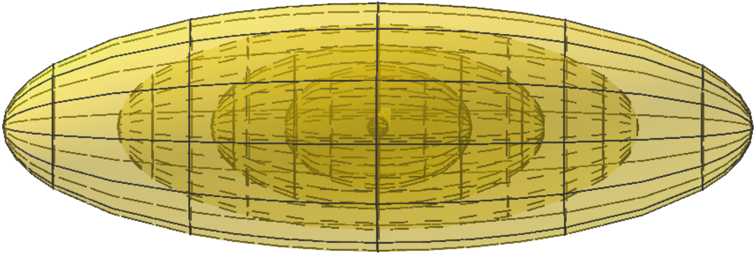

Figure: Snapshots of the ancient ovaloid of White, X.-J. Wang, Haslhofer–Hershkovits and Angenent–Daskalopoulos–Šešum.

Ancient pancakes

Theorem (X.-J. Wang, Bourni–L.–Tinaglia) *There exists an $O(1) \times O(n)$-invariant ancient pancake $\{\Pi_t^n\}_{t \in (-\infty, 0)}$ in $\mathbb{R}^{n+1}$ for each n.*

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Note that there can be no examples with $O(k)\times O(n-k)$-symmetry when $k>1$ (the shrinking cylinder is a barrier).

Ancient pancakes

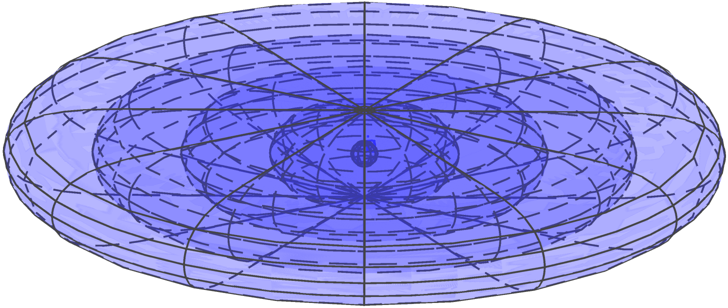

Figure: Snapshots of the rotationally symmetric ancient pancake.

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Remarks: 'Existence' is harder than for ovaloids, since 'scaling is not allowed'. Uniqueness is (hard, but) easier than for ovaloids since 'scaling is unnecessary' and (surprisingly) no analysis of the 'intermediate region' is needed.