An Introduction to Ancient Ricci Flows and $3D$ Gradient Shrinkers

Timothy Buttsworth

Ricci Flow

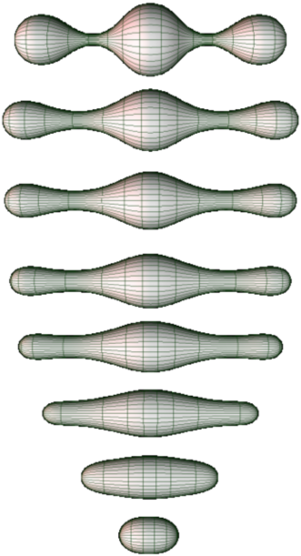
Definition (Ricci Flow)

A one-parameter family of smooth Riemannian metrics $\{g_t\}_{t \in I}$ on a manifold M is said to be a *Ricci flow* if for all t in the interval I, we have

$$\frac{\partial g_t}{\partial t} = -2\mathrm{Ric}(g_t),$$

where $\mathrm{Ric}(g_t)$ is the Ricci curvature of g_t.

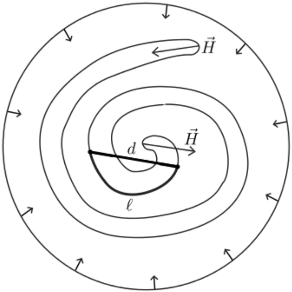
Ricci Flow



https://en.wikipedia.org/wiki/Ricci_flow

Ricci Flow in 1D

In mean curvature flow, the following curve unravels before collapsing to a single point. On the other hand, Ricci curvature is *intrinsic*, so the curve is unaffected by Ricci flow!



Klaus Ecker, *Regularity Theory for Mean Curvature Flow*

Ricci Flow in 2D

2D Ricci Flow is Conformal

In 2D, the Ricci flow preserves conformal class. Therefore, if our initial metric is (M, g_0), then our solution is $g(t) = u(t,x)g_0$ for some function $u : I \times M \to \mathbb{R}$, and

$$\frac{\partial u}{\partial t} = \Delta_{g_0} \log(u) - S(g_0), \tag{1}$$

where Δ_{g_0} is the Laplace-Beltrami operator, and $S(g_0)$ is the scalar curvature of g_0.

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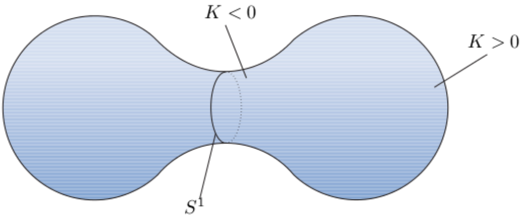

Peter Topping, *Lectures on the Ricci Flow*

Ancient Ricci Flow

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Evolution of Scalar Curvature

If $\{g_t\}_{t \in I}$ is a Ricci flow and $S(g_t)$ is the scalar curvature of g_t, then

$$\frac{\partial S}{\partial t} = \Delta_{g_t} S + 2 \left|\mathrm{Ric}\right|^2 .$$

Ancient Ricci Flow

Theorem (Chen 2009)

If $\{g_t\}_{t\in I}$ is a complete ancient Ricci flow, then $S(g_t) \geq 0$ for each $t \in I$.

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If $\{g_t\}_{t \in I}$ is a complete ancient Ricci flow on a three-dimensional manifold, then it has non-negative sectional curvature.

Gradient Shrinking Ricci Solitons

Definition (Ricci Solitons)

A smooth and complete Riemannian manifold (M, g) is said to be a *gradient shrinking Ricci soliton* if there is a smooth function $f : M \to \mathbb{R}$ so that

$$\mathrm{Ric}(g) + \mathrm{Hess}_g(f) = \frac{\lambda}{2} g$$

for some constant $\lambda > 0$. If $\lambda = 0$, the Riemannian manifold is said to be a *steady Ricci soliton*.

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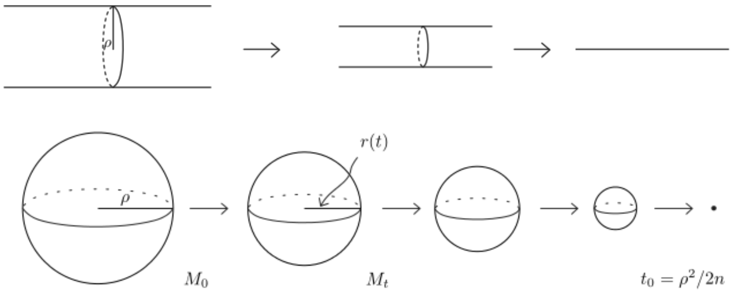
Behaviour in Ricci flow

If g_0 is a gradient shrinking or steady Ricci soliton, then $g(t) = \sigma(t)\phi(t)^* g_0$ is an ancient Ricci flow, with $\sigma(t) = 1 - \lambda t$, and $\phi(t)$ a diffeomorphism of M generated by $\nabla_{g_0} f$.
Shrinking Ricci solitons arise as singularity models for the Ricci flow, so understanding them becomes important in, for example, the Poincaré conjecture.

Gradient Shrinking Ricci Solitons



Gradient Steady Ricci Solitons

Cigar Soliton

The metric $g = \frac{dx^2+dy^2}{1+x^2+y^2}$ is a steady Ricci soliton on $\mathbb{R}^2$. The diffeomorphism of evolution is generated by $\nabla f = -2(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})$.

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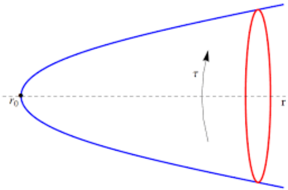

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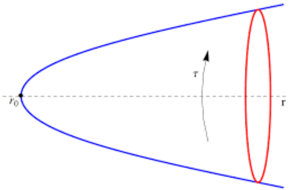

Bryant Soliton

The analogue of the Cigar soliton on $\mathbb{R}^n$ ($n \geq 3$) is the *Bryant soliton*. It is also asymptotically cylindrical, but is more difficult to construct because the fibers $\mathbb{S}^{n-1}$ now have intrinsic curvature.

$3D$ Solitons: The Compact, Positive Curvature Case

Theorem (Hamilton 1982)

If (M, g) is a compact 3-dimensional Riemannian manifold with positive Ricci curvature, then the Ricci flow terminates in finite time. After renormalising, the metric converges to (a quotient of) the round sphere.

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This shows that the round sphere is the only 3-dimensional compact Ricci soliton.

Growth of the Potential Function

Theorem (Cao-Zhao 2010)

Suppose we have a shrinking GRS with $\lambda = 1$. *Then there are positive constants* c_1, c_2 *and a point* $x_0 \in M$ *so that for* $d(x_0, x)$ *large,*

$$\frac{1}{4}\left(d(x_0, x) - c_1\right)^2 \leq f(x) \leq \frac{1}{4}\left(d(x_0, x) + c_2\right)^2$$

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Remark (The Gaussian Shrinker)

One example of a shrinker with $\lambda = 1$ is $M = \mathbb{R}^n$, g the standard Euclidean metric, and $f(x) = \frac{|x|^2}{4}$, so the coefficient of $\frac{1}{4}$ is optimal.

Growth of the Potential Function

Proof of upper bound

- If $S(g)$ is the scalar curvature of g, then $S + |\nabla f|^2 - f = C_0$ is constant on M (if M is connected), so by adding a constant to f, we can assume that $C_0 = 0$.

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- Since $S(g) \geq 0$, we obtain that $|\nabla f|^2 \leq f$.
- Integrating gives $f(x) \leq \frac{1}{4}\left(d(x_0, x) + 2\sqrt{f(x_0)}\right)^2$.

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Proof of lower bound (part 1)

- Consider any minimising arc-length geodesic $\gamma : [0, s_0] \to M$ with $\gamma(0) = x_0$, $\gamma(s_0) = y$ and $s_0 > 2$.

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- Then by the variation formula for energy,

$$\int_0^{s_0} \phi^2 \mathrm{Ric}(\gamma', \gamma') \leq (n-1) \int_0^{s_0} (\phi')^2 = 2n - 2.$$

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- Then since $\mathrm{Ric}(\gamma', \gamma') = \frac{1}{2} - \nabla_{\gamma'} \nabla_{\gamma'} f$, we find

$$\begin{aligned} \frac{d(x_0, y)}{2} + \frac{4}{3} - 2n &\leq \int_0^{s_0} \phi^2 \nabla_{\gamma'} \nabla_{\gamma'} f \\ &\leq 1 + \sqrt{f(x_0)} + \sqrt{f(y)}. \end{aligned}$$

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- We can therefore choose x_0 to be the minimiser of f.

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- The lower bound then follows from our previous estimate

$$\frac{d(x_0, y)}{2} + \frac{4}{3} - 2n \leq 1 + \sqrt{f(x_0)} + \sqrt{f(y)}.$$

Volume Growth

Lemma (Cao-Zhao 2010)

If $D(r)$ is the set of x with $f(x) \leq \frac{r^2}{4}$, then $\int_{D(r)} S \leq \frac{n}{2} Vol(D(r))$.

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- Let $V(r) = \int_{D(r)} 1$.

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- Then $nV(r) - 2\int_{D(r)} S = 2\int_{D(r)} \Delta f = 2\int_{\partial D(r)} \nabla f \cdot \frac{\nabla f}{|\nabla f|} \geq 0$.

□

The Noncompact, Positive Curvature Case

Theorem (Munteanu- Wang 2015)

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Remark

We can assume without loss of generality that $\lambda = 1$. We may also add a constant to f so that $S + |\nabla f|^2 = f$ everywhere.

The Noncompact, Positive Curvature Case

Lemma (A lower bound on Ricci curvature)

Let $\lambda : M \to \mathbb{R}^+$ be the function which returns the smallest Ricci eigenvalue. Then there exists a $0 < b < 1$ so that, for $d(x, x_0)$ large, $\lambda(x) \geq \frac{b}{f}$.

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- GRS equation for Ricci curvature is $\Delta_f R_{ij} = R_{ij} - 2R_{ikjl}R_{kl}$, where $\Delta_f \cdot = \Delta \cdot - g(\nabla f, \nabla \cdot)$.

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- For $r(x) > r_0$, we have $\Delta_f u \leq u$, where $u = \lambda - \frac{a}{f} - \frac{an}{f^2}$, $a = \inf_{\partial B_{r_0}(p)} \lambda > 0$.

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- Provided r_0 is made larger if necessary, we have $u > 0$ on $\partial B_{r_0}(p)$, and u is asymptotically non-negative as $r \to \infty$. The maximum principle then implies that $u \geq 0$ for $r(x) > r_0$.

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- If $\nabla f \neq 0$ then we can integrate in this direction. Using $\mathrm{Ric} \geq \frac{b}{f}$ gives the result.
- Since $|\nabla f|^2 = f - S$, the only points with $\nabla f = 0$ are either close to p, or have $S \geq b \ln(f)$ anyway.

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The Noncompact, Positive Curvature Case

Proof of Theorem

- Recall that $S(x) \geq b \ln(f(x))$, $f(x) \sim \frac{d(x,x_0)^2}{4}$ for large $d(x, x_0)$. Therefore, if we are non-compact, then for any q with $d(x_0, q) = \frac{3r}{4}$ ($r > 0$ large), we have

$$\int_{B_{x_0}(r)} S \geq \int_{B_q(\frac{r}{4})} S \geq b \ln\left(\frac{r}{4} - c\right)^2 \mathrm{Vol}(B_q(\frac{r}{4})).$$

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- Also recall that the average value for S on $D(r) = \{x : f(x) \leq \frac{r^2}{4}\}$ is bounded by $\frac{n}{2}$, so

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- However, we also have $\mathrm{Vol}(B_q(\frac{r}{4})) \geq c(n)\mathrm{Vol}(B_{x_0}(r))$ by Bishop-Gromov volume comparison, which is a contradiction.

The Case of Non-Negative Curvature

Hamilton-Ivey Pinching

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- In three-dimensions, $\mathcal{R}^2 + \mathcal{R}^{\#}$ is positive semi-definite whenever g_t has non-negative sectional curvature.
- In fact, one can show from this equation that, under the Ricci flow, positive sectional curvatures dominate negative sectional curvatures.

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Theorem (Hamilton's Splitting Theorem)

A simply-connected, non-negatively curved 3-dimensional GRS M with a Ricci eigenvector of 0 somewhere must be of the form $\mathbb{R} \times N$ for some 2-dimensional GRS N.

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- Suppose $(x_0, v) \in TM$ is a point with $\text{Ric}(v, v) = 0$. Extend v to a vector field on M by parallel transporting it along radial geodesics from x_0.
- The resulting function $\text{Ric}(v, v)$ is non-negative, and has a minimum of zero at x_0; the maximum principle implies that this function is uniformly zero.
- For each $x \in M$, split $T_x M$ into $v \oplus v^\perp$, and this decomposition is invariant under the holonomy group. Apply the de Rham decomposition Theorem.

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- If we are compact and have positive curvature, Hamilton's rounding Theorem implies that we are $\mathbb{S}^2$ or $\mathbb{RP}^2$.

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- We cannot replace $\mathbb{R}^3$ with quotients because it will not be a gradient shrinking soliton anymore.

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- The presented proof is essentially without holes, except for:
  - Hamilton's rounding Theorem (Ricci flow turns positively curved 2 and 3-manifolds to spheres) and
  - Chen's local pinching estimates (ancient Ricci flows have non-negative scalar curvature, and ancient $3D$ Ricci flows have non-negative sectional curvature).