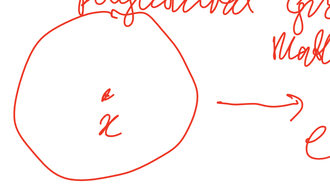
Peter Li, "harmonic sections of polynomial growth" Math. Res. Lett '97.



Harmonic functions

1. The Euclidean case

Mean value formula

$$u(x) = \fint_{B_r(x)} u = \fint_{\partial B_r(x)} u$$

Gradient estimate:

$$Du(x)\cdot e = \frac{1}{\omega_n r^n}\int_{\partial B_r(x)} u\,\nu\cdot e = \frac{n}{r}\fint_{\partial B_r(x)} u\,\nu\cdot e$$

- If $|u| \leq M$ on $B_r(x)$ then
  $$|Du(x)| \leq \frac{n}{r} M$$
- If $u \geq 0$ on $B_r(x)$ then
  $$|Du(x)| = \frac{n}{r}\fint_{\partial B_r} u\,\nu\cdot e \leq \frac{n}{r}\fint_{\partial B_r} u = \frac{n}{r} u(x)$$
  $$\Rightarrow \frac{|Du(x)|}{u(x)} \leq \frac{n}{r}$$

$\Rightarrow$ Liouville theorems for bounded or non-negative functions

Theorem: Harmonic functions on $\mathbb{R}^n$ of polynomial growth are polynomials.

i.e. $\mathcal{H}_p(\mathbb{R}^n) = \{u \in C^\infty(\mathbb{R}^n) \mid \Delta u = 0,\ |u(x)| \leq C(1+|x|)^p\}$

Key step:

Lemma: $u \in \mathcal{H}_p(\mathbb{R}^n) \Rightarrow \begin{cases} \frac{\partial u}{\partial x^i} \in \mathcal{H}_{p-1}(\mathbb{R}^n) & \text{if } p \geq 1 \\ \frac{\partial u}{\partial x^i} \equiv 0 & \text{if } p < 1 \end{cases}$

Proof:

$x \in B_r(0) \Rightarrow |Du(x)| \leq \frac{n}{r} \sup_{B_r(x)} |u| \leq \frac{n}{r} C(1+r+|x|)^p$

Choose $r = |x|$

$\Rightarrow |Du(x)| \leq \frac{n}{r} C(1+2r)^p$

$\leq C(1+r)^{p-1} \quad (r > 1).$

$\Rightarrow \frac{\partial u}{\partial x^i} \in \mathcal{H}_{p-1}(\mathbb{R}^n).$

Corollary: $u \in \mathcal{H}_p(\mathbb{R}^n) \Rightarrow \frac{\partial^k u}{\partial x^{i_1} \cdots \partial x^{i_k}} \equiv 0$ if $k > p$.

$\Rightarrow u$ is a polynomial $\square$.

$$\dim \mathcal{H}_p(\mathbb{R}^n) = \binom{n+p-1}{n-1} + \binom{n+p-2}{n-1} \sim \frac{2}{(n-1)!} p^{n-1}$$

* Generalize to manifolds with $Ric \geq 0$

Theorem: If (M^n, g) is complete (non-compact) with $Ric(g) \geq 0$ then

$$\dim \mathcal{H}_p(M) \leq C(n) p^{n-1}$$

~~Hessian~~ Laplacian comparison theorem: If $Ric \geq 0$

$$\rho(x) := d(x, x_0) \implies \Delta \rho^2 \leq 2n \quad \left(\Delta\rho \leq \frac{n-1}{\rho}\right)$$

- holds where ρ is smooth (away from cut locus)

- Barrier sense:

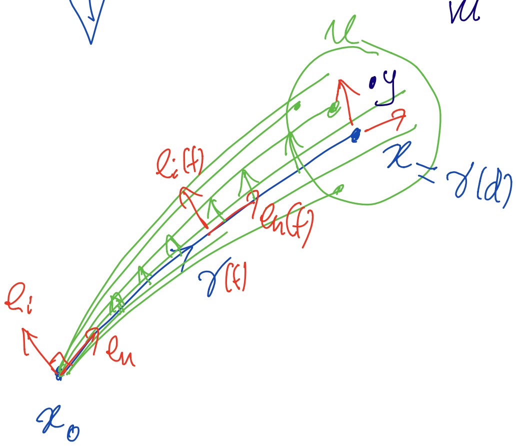
Given any $x \in M$ there is $\tilde\rho$ smooth defined in a nbhd of x with

$$\tilde\rho \geq \rho$$
$$\tilde\rho(x) = \rho(x)$$
$$\Delta\tilde\rho(x) \leq \frac{n-1}{\rho(x)}$$

- Distributional sense:

If $\varphi \in C_c^\infty(M)$, $\varphi \geq 0$, then

$$\int_M \Delta\varphi\, \rho^2 = -\int_M \nabla\varphi \cdot \nabla\rho^2 \leq 2n \int_M \varphi$$



$$\gamma(s_1, \dots, s_n, t) = \exp_{\gamma(t(d+s_n))}\left(\sum_{i=1}^{n-1} t\, s_i e_i(t(d+s_n))\right)$$

$$\gamma(s_1, \dots, s_n, 1) = \Psi(s_1, \dots, s_n)$$

is a diffeomorphism to a nbhd of x.

$$\tilde\rho(y) := L[\underbrace{\gamma(\Psi^{-1}(y), t)}_{\text{curve from } x_0 \text{ to } y}] \geq d(x_0, y) = \rho(y)$$

$$\tilde\rho(x) = \rho(x).$$

$$\Delta\tilde\rho(x) = \sum_{i=1}^{n-1} \int_0^1 d\left|\nabla_t \frac{\partial\gamma}{\partial s_i}\right|^2 - dR\left(\gamma', \frac{\partial\gamma}{\partial s_i}, \gamma', \frac{\partial\gamma}{\partial s_i}\right) dt$$

$$\frac{\partial \gamma'}{\partial s_i} = t e_i'$$

$$\nabla_t \frac{\partial \gamma}{\partial s_i} = e_i$$

$$= \frac{n-1}{d} - d \int_0^1 t Rc(\gamma', e_i; \gamma', e_i)\, dt$$

$$\leq \frac{n-1}{\rho(x)} .$$

Distributional sense:

$$-\int_M D\varphi \cdot D\rho^2 = -\lim_{\varepsilon \to 0} \int_{M_\varepsilon} D\varphi \cdot D\rho^2$$

$M = \exp_{x_0}(\mathcal{E})$ $\quad$ $\mathcal{E}$ starshaped in $T_{x_0}M$ locally Lipschitz bdry.

$M_\varepsilon = \exp_{x_0}((1-\varepsilon)\mathcal{E}) \Rightarrow \rho$ is smooth on M_ε

$$\int_{M_\varepsilon} D\varphi \cdot D\rho^2 = \int_{\partial M_\varepsilon} \varphi D_\nu \rho^2 - \int_{M_\varepsilon} \varphi \Delta\rho^2$$

$$= \underbrace{\int_{\partial M_\varepsilon} 2 \, d\varphi}_{\geq 0.} \qquad \geq 2n \int_{M_\varepsilon} \varphi$$

$\lim_{\varepsilon \to 0}$: $\quad \int_{M_\varepsilon} \varphi \to \int_M \varphi$

Consequences:

(i) Volume growth (Bishop–Gromov).

$$2n|B_r(x)| \geq \int_{B_r(x)} \Delta \rho^2 = \int_{\partial B_r(x)} D_\nu \rho^2 = 2r|\partial B_r|$$

$$\Longrightarrow \quad \frac{d}{dr}\left(\frac{1}{r^n}|B_r|\right) = -\frac{n}{r^{n+1}}|B_r| + \frac{1}{r^n}|\partial B_r|$$

$$\leq -\frac{n}{r^{n+1}}|B_r| + \frac{n}{r}\frac{1}{r^n}|B_r|$$

$$\leq 0.$$

$$\Longrightarrow \quad |B_R(x)| \leq \left(\frac{R}{r}\right)^n |B_r(x)| \quad \text{if } 0<r<R.$$

(ii). <u>Poincaré inequality</u> (Li-Schoen)

Assume (M^n, g) complete, non-compact, $Rc(g) \geq 0$.
Then for $\varphi \in W_0^{1,2}(B_r(x_0))$

$$\int_{B_r(x_0)} |\nabla \varphi|^2 \geq \frac{C(n)}{r^2}\int_{B_r(x_0)} \varphi^2.$$

Proof: WLOG $\varphi \in C_c^\infty(B_r(x_0))$.

For $\eta \in C^\infty(B_r(x_0))$ with $\eta > 0$

$$0 \leq \int_{B_r} \eta^2 \left|\nabla\left(\frac{\varphi}{\eta}\right)\right|^2 = \int_{B_r} \eta^2 \left|\frac{\nabla\varphi}{\eta} - \frac{\varphi}{\eta^2}\nabla\eta\right|^2 = \int |\nabla\varphi|^2 - \overbrace{2\varphi\nabla\varphi}^{\nabla\varphi^2}\frac{\nabla\eta}{\eta} + \frac{|\nabla\eta|^2}{\eta^2}\varphi^2$$

$$= \int |\nabla\varphi|^2 + \varphi^2\left(\frac{\Delta\eta}{\eta} - \cancel{\frac{|\nabla\eta|^2}{\eta^2}} + \cancel{\frac{|\nabla\eta|^2}{\eta^2}}\right)$$

$$= \int |\nabla\varphi|^2 + \varphi^2\frac{\Delta\eta}{\eta}$$

Choose η s.t. $\frac{\Delta\eta}{\eta}$ is uniformly negative.

For any $x_k \notin B_r(x)$, let $\rho_k = d(\cdot, x_k) - d(x_0, x_k) + r + \varepsilon$

$d_k = d(\cdot, x_k) \Rightarrow \Delta d_k \leq \frac{n-1}{d_k}$ $\Longrightarrow$ $\Delta\rho_k \leq \frac{n-1}{d_k}$ $|\nabla\rho_k| \leq 1$

Now let $x_k \to \infty$ $\Rightarrow \Delta\rho \le 0$ since ρ [illegible]

(Busemann function)

$$2r+\varepsilon \ge \rho \ge \varepsilon \quad \text{on } B_r(x)$$

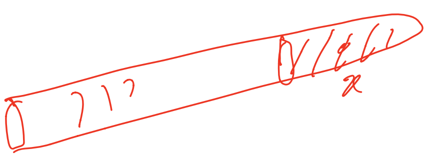
$$\Delta\rho \le 0.$$

$$\eta = \sin\left(\frac{\pi}{2(2r+\varepsilon)}\rho\right)$$

$$D_i\eta = \frac{\pi}{2(2r+\varepsilon)}\cos(\quad)D_i\rho$$

$$\Delta\eta = \underbrace{\frac{\pi}{\quad}\cos(\quad)}_{>0}\Delta\rho$$

$$+ -\frac{\pi^2}{4(2r+\varepsilon)^2}\sin(\quad)|\nabla\rho|^2$$



$$\le -\frac{\pi^2}{4(2r+\varepsilon)^2}\eta$$

$$\frac{\Delta\eta}{\eta} \le -\frac{\pi^2}{4(2r+\varepsilon)^2}$$

$$\Longrightarrow \int_{B_r(x)} |\nabla\varphi|^2 \le \frac{\pi^2}{16r^2}\int_{B_r(x)}\varphi^2.$$

Next: • Yau gradient estimate

• Mean value inequality:

If $v \ge 0$, $\Delta v \ge 0$

$$\Longrightarrow \quad v(x_0) \le C(n) \fint_{B_r(x_0)} v$$