(M^n, g), $Ric \geq 0$ complete, non-compact.

1. Laplacian comparison $\Delta\rho^2 \leq 2n$ $\rho(x) = d(x, x_0)$
2. Volume comparison $|B_R(x_0)| \leq \left(\frac{R}{r}\right)^n |B_r(x_0)|$ for $0<r<R$.
3. $\int_{B_r(x_0)} |\nabla\varphi|^2 \geq \frac{C(n)}{r^2}\int_{B_r(x_0)} \varphi^2$ for $\varphi|_{\partial B_r(x_0)} = 0$.

Yau gradient estimate

Claim: If $u>0$, $\Delta u = 0$ on $B_r(x_0) \subset M$

then $(r^2 - \rho(x)^2)|\nabla \log u(x)| \leq C(n) r$ on $B_r(x_0)$.

Proof: WLOG $r=1$. $\Delta u = 0$, $u>0$ $v = \log u$.

$$\Delta v = \nabla_i\left(\frac{\nabla_i u}{u}\right) = \cancel{\frac{\Delta u}{u}} - \left|\frac{\nabla u}{u}\right|^2 = -|\nabla v|^2. \quad (1)$$

let $F = |\nabla v|^2$.

Diff (1) $\Rightarrow$

$$0 = \nabla_i(\Delta v + F)$$
$$= \Delta\nabla_i v - R_i^p \nabla_p u + \nabla_i F$$

$$\Rightarrow 0 = \Delta F - 2|\nabla^2 v|^2 - 2Rc(\nabla v, \nabla v) + 2\nabla v \cdot \nabla F$$
$$\leq \Delta F - \frac{2}{n}(\Delta v)^2 + 2\sqrt{F}|\nabla F|$$
$$= \Delta F - \frac{2}{n}F^2 + 2\sqrt{F}|\nabla F|. \quad (2).$$

Write $\varphi = 1-\rho^2$. $\rho = d(\cdot, x_0)$.

let $G = \varphi^2 F$. On $\partial B_1(x_0)$ $G=0$

$\Rightarrow$ G attains an interior max. at $\bar{x} \in B_1(x_0)$.

$|\nabla F| = 2\left|\frac{\nabla\varphi}{\varphi}\right| F$

At $\bar{x}$, $0 = \nabla_i G = \varphi^2 \nabla_i F + 2\varphi\nabla_i\varphi F$

$$= \varphi^2\left(\nabla_i F + 2\frac{\nabla_i \varphi}{\varphi^3} G\right)$$

At $\bar{x}$, $\quad 0 \geq \Delta G = \varphi^2\left(\Delta F + \left(2\frac{\Delta\varphi}{\varphi^3} - 6\frac{|\nabla\varphi|^2}{\varphi^4}\right)\varphi^2 F\right)$

$$\geq \varphi^2\left(\frac{2}{n}F^2 - 2\sqrt{F}\cdot 2\frac{|\nabla\varphi|}{\varphi}F + \left(2\frac{\Delta\varphi}{\varphi} - 6\frac{|\nabla\varphi|^2}{\varphi^2}\right)F\right)$$

$$= F\left(\frac{2}{n}G - 4\sqrt{G}\,|\nabla\varphi| + 2\varphi\Delta\varphi - 6|\nabla\varphi|^2\right)$$

$$\geq F\left(\frac{1}{n}G - C|\nabla\varphi|^2 + 2\varphi\Delta\varphi\right)$$

$\varphi = 1-\rho^2$, $\quad \nabla_0\varphi = -2\rho|\nabla\rho| \Rightarrow |\nabla\varphi| \leq 4\rho^2 \leq 4$

$$\Delta\varphi = -\Delta\rho^2 \geq -2n$$

$$\Rightarrow \quad G \leq C(n). \qquad \square.$$

Harnack inequality:

If $u>0$, $\Delta u = 0$ on $B_r(x_0) \subset M$,
then

$$\sup_{B_{r/2}(x_0)} u \leq C(n) \inf_{B_{r/2}(x_0)} u.$$

Proof: WLOG $r=1$.

On $B_{1/2}(x_0)$, $|\nabla \log u| \leq C(n)$.

$$\Rightarrow \quad \frac{u(y)}{u(x)} \leq e^{C(n) d(x,y)} \leq e^{C(n)}$$

Mean value inequality (Li-Schoen)

If v is a non-negative subharmonic function on $B_r(x_0)$, then $\qquad \Delta v \geq 0.$

$$v(x_0) \leq C(n) \fint_{B_r(x_0)} v.$$

Prove: The case where $v = u^2$, $\Delta u = 0$.

$$\Delta v = 2|\nabla u|^2 \geq 0.$$

Case 1: If h is harmonic then
$h \geq 0$

Harnack:

$$h(x_0) \leq \sup_{B_{r/2}(x_0)} h \leq C(n) \inf_{B_{r/2}(x_0)} h$$

$$\leq C(n) \fint_{B_{r/2}(x_0)} h$$

$$= C(n) \frac{\int_{B_{r/2}(x_0)} h}{|B_{r/2}(x_0)|}$$

$$\leq C(n) \frac{\int_{B_r(x_0)} h}{|B_r(x_0)|} \frac{|B_r(x_0)|}{|B_{r/2}(x_0)|}$$

$$\leq 2^n C(n) \fint_{B_r(x_0)} h.$$

Case 2: $v = u^2$, $\Delta u = 0$.

Let h be the harmonic function on $B_{1/2}$ with $h\big|_{\partial B_{1/2}} = |u|$.

$|u|$ is subharmonic $\Rightarrow$ $|u|(x_0) \leq h(x_0)$

$$u^2(x_0) \leq h^2(x_0) \leq C(n) \Big(\fint_{B_{1/4}(x_0)} h \Big)^2$$

$$\leq C(n) \fint_{B_{1/4}(x_0)} h^2$$

Write $\int_{B_{1/2}} h^2 = \int_{B_{1/2}} (h-|u|+|u|)^2 \leq 2\int_{B_{1/2}} (h-|u|)^2 + 2\int_{B_{1/2}} u^2$

0 on $\partial B_{1/2}$.

$$\int_{B_{1/2}} (h-|u|)^2 \underset{\text{Poincaré ineq.}}{\leq} C(n)\int_{B_{1/2}} |\nabla h - \nabla |u||^2 \leq C\int_{B_{1/2}} |\nabla h|^2 + |\nabla u|^2$$

$$\cdot\, h \text{ harmonic} \Rightarrow \int_{B_{1/2}} |\nabla h|^2 \leq \int_{B_{1/2}} |\nabla |u||^2 \leq \int_{B_{1/2}} |\nabla u|^2$$

let $\Phi \in C_c^\infty(B_1)$, $\Phi \equiv 1$ on $B_{1/2}$, $|\nabla \Phi| \leq C$.

$$\int_{B_1} \Phi^2 |\nabla u|^2 = \cancel{-\int_{B_1} \Phi^2 u \Delta u} - \int_{B_1} 2\Phi u \nabla\Phi \cdot \nabla u$$

$$= 2\left(\int_{B_1} \Phi^2 |\nabla u|^2\right)^{1/2} \left(\int_{B_1} u^2 |\nabla \Phi|^2\right)^{1/2}$$

$$\Rightarrow \int_{B_1} \Phi^2 |\nabla u|^2 \leq 4\int_{B_1} u^2 |\nabla \Phi|^2 \leq C\int_{B_1} u^2$$

$$\vee\!/$$

$$\int_{B_{1/2}} |\nabla u|^2$$

$$\int_{B_{1/2}} h^2 \leq C\int_{B_1} u^2$$

$$\Rightarrow u^2(x_0) \leq \frac{C}{|B_{1/2}(x_0)|}\int_{B_1} u^2 \leq 2^n C ⨍_{B_1} u^2$$

<u>Main Theorem:</u>

$$\dim \mathcal{H}_p(M) \leq C(n)\, p^{n-1}$$

Colding-Minicozzi '96
Peter Li '96.

where $\mathcal{H}_p(M) = \{ u \in C^\infty(M) \mid \Delta u = 0, \ |u(x)| \leq C(1+d(x,x_0))^p \}$

Lemma 1: Let K be any finite-dimensional space of $\mathcal{H}(M) = \{u \in C^\infty(M) \mid \Delta u = 0\}$.
Let $\{u_i\}_{i=1}^k$ be any o/n basis for K w.r.t. $L^2(B_r(x))$.
Then for any $\varepsilon \in (0, \frac{1}{2}]$

$$\int_{B_{r(1-\varepsilon)}(x)} \sum_{i=1}^k u_i^2 \le C(n)\,\varepsilon^{-(n-1)}$$

* RHS. is independent of k.

* $\int_{B_r(x)} u_i u_j = \delta_{ij}$

$\Rightarrow \int_{B_r(x)} \sum_{i=1}^k u_i^2 = k.$

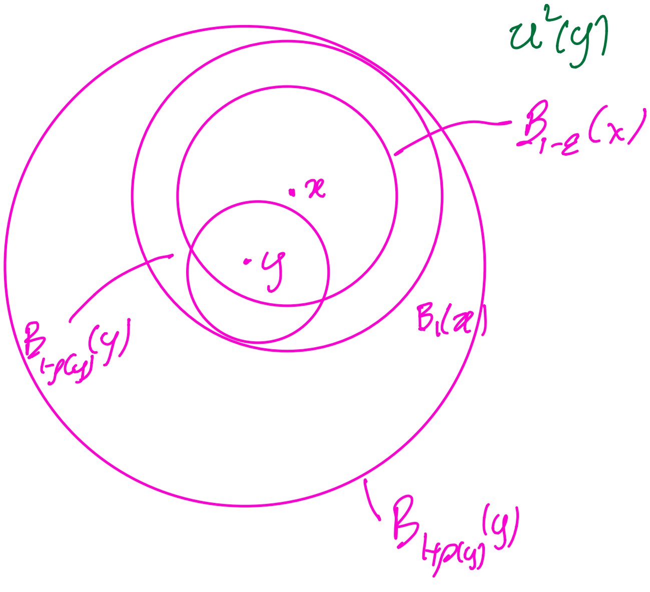
Proof: WLOG $r=1$.

If $y \in B_{1-\varepsilon}(x)$, choose a rotation $\mathcal{O}$ of $\mathbb{R}^k$ such that

$$\begin{pmatrix}\mathcal{O}_1^i u_i(y) \\ \vdots \end{pmatrix} = \mathcal{O}\begin{pmatrix}u_1(y)\\ \vdots \\ u_k(y)\end{pmatrix} = \sqrt{\sum_{i=1}^k u_i^2(y)}\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix}$$

Let $u = \sum_i \mathcal{O}_1^i u_i$ harmonic!

M.V.I.



$$u^2(y) \le C \fint_{B_{1-\rho(y)}(y)} u^2 = C\,\frac{\int_{B_{1-\rho(y)}(y)} u^2}{|B_{1-\rho(y)}(y)|}$$

$$\le C\,\frac{\int_{B_1(x)} u^2}{|B_1(x)|}\,\frac{|B_1(x)|}{|B_{1-\rho(y)}(y)|}$$

$$\le C\,\frac{|B_{1+\rho}(y)|}{|B_{1-\rho}(y)|}\fint_{B_1(x)} u^2$$

$$\le C\left(\frac{1+\rho}{1-\rho}\right)^n \fint_{B_1(x)} u^2$$

Note $\int_{B_1(x)} u^2 = \int_{B_1(x)} \mathcal{O}_1^i \mathcal{O}_1^j u_i u_j = \sum_{i=1}^k (\mathcal{O}_1^i)^2 = 1$ since $\{u_i\}$ o/n in $L^2(B_1)$.

Easy estimate:

On $B_{1-\varepsilon}(x)$, $\frac{1+\rho}{1-\rho} \leq \frac{2}{\varepsilon}$

$$\sum_{i=1}^{k} u_i^2(y) = u^2(y) \leq \frac{C\varepsilon^{-n}}{|B_1(x)|}$$

$$\int_{B_{1-\varepsilon}(x)} \sum_{i=1}^{k} u_i^2 < C\varepsilon^{-n} \frac{|B_{1-\varepsilon}|}{|B_1|} \leq 1$$