Lemma 1: Let K be any finite-dimensional space of $\mathcal{H}(M) = \{ u \in C^\infty(M) \mid \Delta u = 0 \}$.
Let $\{u_i\}_{i=1}^k$ be any o/n basis for K w.r.t. $L^2(B_r(x))$.
Then for any $\varepsilon \in (0, \frac{1}{2}]$

$$\int_{B_{r(1-\varepsilon)}(x)} \sum_{i=1}^k u_i^2 \le C(n)\, \varepsilon^{-(n-1)}$$

Recall: wlog $r = 1$

If $y \in B_{1-\varepsilon}(x)$ $\qquad u^2(y) = \sum_{i=1}^k u_i^2(y)$, relative in K.

$$\Rightarrow \sum_{i=1}^n u_i^2(y) \le C(n) \left(\frac{1+\rho(y)}{1-\rho(y)} \right)^n ⨍_{B_1} u^2 = \frac{1}{|B_1|}$$

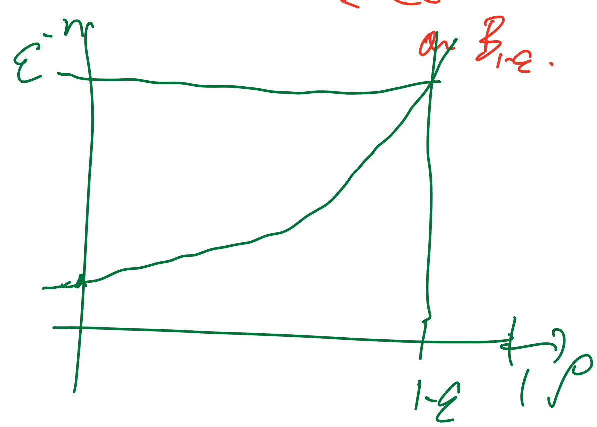
$\rho = d(\cdot, x)$.

$\int_{B_1} u^2 = 1$

$$\int_{B_{(1-\varepsilon)}(x)} \sum_{i=1}^k u_i^2 \le \frac{C(n)}{|B_1(x)|} \int_{B_{(1-\varepsilon)}(x)} \left(\frac{1+\rho(y)}{1-\rho(y)} \right)^n$$

$\le C \varepsilon^{-n}$ on $B_{1-\varepsilon}$.

Better estimate:



$$\int_{B_{1-\varepsilon}\setminus B_{1/2}} (1-\rho)^{-n} = \int_{B_{1-\varepsilon}\setminus B_{1/2}} (1-\rho)^{-n} |\nabla\rho|^2 = \frac{1}{n-1}\int_{B_{1-\varepsilon}\setminus B_{1/2}} \nabla\left((1-\rho)^{-(n-1)} - \varepsilon^{-(n-1)}\right)\cdot\nabla\rho$$

$$= -\frac{1}{n-1}\int_{\partial B_{1/2}} \left[(1-\rho)^{-(n-1)} - \varepsilon^{-(n-1)}\right]$$

$$- \frac{1}{n-1}\int_{B_{1/\varepsilon}\setminus B_{1/2}} \left((1-\rho)^{-(n-1)} - \varepsilon^{-(n-1)}\right)\underbrace{\Delta\rho}_{\leq \frac{n-1}{\rho} \leq 2(n-1).}$$

$$\leq C\varepsilon^{-(n-1)}$$

<u>Lemma 2</u>: Let K be a finite-dimensional subspace of $\mathcal{H}_p(M) = \{u\in C^\infty(M) \mid \Delta u = 0,\ |u(y)| \leq C(1+d(y,x_0))^p\}$.

Then for any $x\in M$, $\varepsilon\in(0,\frac{1}{2}]$, $r_0>0$, $\delta>0$, there exists $r>r_0$ such that if $\{u_i\}_{i=1}^k$ is an o/n basis for K w.r.t. $L^2(B_r(x))$, then

$$\sum_{i=1}^k \int_{B_{(1-\varepsilon)r}(x)} |u_i|^2 \geq k(1-\varepsilon)^{2p+n+\delta} \qquad \left(\int_{B_r(x)} \sum_{i=1}^k u_i^2 = k\right)$$

$$= (1-\varepsilon)^{2p+n+\delta}\int_{B_r(x)} \sum_{i=1}^k u_i^2$$

<u>Proof</u>: Fix $x\in M$, $\varepsilon\in(0,\frac{1}{2}]$, $r_0>0$, $\delta>0$.

Define $r_\alpha = r_0(1-\varepsilon)^{-\alpha}$ $\quad \alpha = 1,2,\dots$

Suppose the claimed inequality does not hold for $r = r_\alpha$, $\alpha\in\mathbb{N}$.

Let G_α be the inner product on K coming from $L^2(B_{r_\alpha}(x))$. $\Rightarrow$ If $\{u_i\}_{i=1}^{k}$ is an o/n basis for K w.r.t. $L^2(B_{r_\alpha}(x))$ then

$$(G_\alpha)_{ij} = \delta_{ij}$$

$$(G_{\alpha-1})_{ij} = \int_{B_{(1-\varepsilon)r_\alpha}(x)} u_i u_j \Rightarrow tr(G_\alpha^{-1} \circ G_{\alpha-1}) = \int_{B_{(1-\varepsilon)r_\alpha}} \sum_{i=1}^{k} |u_i|^2$$

$$< k(1-\varepsilon)^{2p+n+\delta}$$

Note $\det(G_\alpha^{-1} \circ G_{\alpha-1}) \leq \left(\frac{1}{k} tr(G_\alpha^{-1} \circ G_{\alpha-1})\right)^k$

$$< (1-\varepsilon)^{(2p+n+\delta)k}$$

$$\Rightarrow \det(G_\alpha^{-1} \circ G_0) = \det(G_\alpha^{-1} \circ G_{\alpha-1}) \det(G_{\alpha-1}^{-1} \circ G_{\alpha-2}) \cdots \det(G_1^{-1} G_0)$$

$$< (1-\varepsilon)^{(2p+n+\delta)k\alpha}$$

Now fix an o/n basis for G_0, $\{u_i\}_{i=1}^{k}$

i.e. $\int_{B_{r_0}(x)} u_i u_j = \delta_{ij}$.

$$\Rightarrow |u_i(y)| \leq C(1+d(x,y))^p \quad (\text{since } u_i \in K \subseteq \mathcal{H}_p)$$

$$\Rightarrow k(\det(G_0^{-1} \circ G_\alpha))^{\frac{1}{k}} \leq tr(G_0^{-1} \circ G_\alpha) = \sum_{i=1}^{k} \int_{B_{r_\alpha}(x)} |u_i|^2$$

$$\vee \qquad\qquad \leq C r_\alpha^{2p} k |B_{r_\alpha}|$$

$$\cancel{k}(1-\varepsilon)^{-(\cancel{2p}+\cancel{n}+\delta)\alpha} \qquad \leq C\cancel{k}|B_{r_0}|\cancel{(1-\varepsilon)^{-\alpha n}} r_0^{2p} \cancel{(1-\varepsilon)^{-2p\alpha}}$$

$$(1-\varepsilon)^{-2\delta} \leq C \quad \text{for all } \alpha > 1.$$

$\downarrow$

∞ as $\alpha \to \infty$. Contradiction

<u>Proof of main theorem:</u>

Let K be any f.d. subspace of $\mathcal{H}_p(M)$.

Let $\varepsilon \in (0, \frac{1}{2}]$. Choose x_0, $r_0 = 1$, $\delta > 0$, $r > r_0$ as in Lemma 2 s.t.

$$\int_{B_{r(1-\varepsilon)}(x)} \sum_{i=1}^{k} |u_i|^2 \geq k(1-\varepsilon)^{2p+n+\delta}.$$

for any o.n. basis $\{u_i\}_{i=1}^{k}$ of K w.r.t. $L^2(B_r(x))$

Lemma 1: $$\int_{B_{r(1-\varepsilon)}(x)} \sum_{i=1}^{k} |u_i|^2 \leq C(n)\varepsilon^{-(n-1)}.$$

$$\Longrightarrow \quad k \leq C(n)\varepsilon^{-(n-1)}(1-\varepsilon)^{-(2p+n+\delta)}$$

Choose $\varepsilon = \frac{1}{2p} \Longrightarrow$

$$k \leq C(n) 2^{n-1} p^{n-1} \left(1-\frac{1}{2p}\right)^{-2p} 2^{n+\delta}$$

$$\leq \tilde{C}(n) p^{n-1}.$$

$\square$.

Remarks:

- Suppose $M^n \subset \mathbb{R}^N$ is a minimal submanifold, with Euclidean volume growth: $V(B_r \cap M) \le C_0 r^n$

Then: The space of harmonic fns. of polynomial growth of order p is bounded by $C(n, C_0) p^{n-1}$.

Idea:

- Michael-Simon prove a mean value inequality for harmonic functions on minimal submanifolds

$$\Delta u = 0, \qquad u^2(x) \le \frac{C}{r^n} \int_{B_r \cap M} u^2$$

- By monotonicity formula

$$\frac{d}{dr}\left(\frac{|B_r \cap M|}{r^n}\right) \ge 0$$

+ Euc. volume growth

$$\Rightarrow \quad |B_r \cap M| \sim r^n.$$

$$\Rightarrow \quad \left|\frac{B_R(x) \cap M}{B_r(x) \cap M}\right| \le C\left(\frac{R}{r}\right)^n.$$

$\Rightarrow$ Proof as before.

<u>Corollary</u>: M is contained in a subspace of $\mathbb{R}^N$ of dimension depending only on n, C_0.

<u>Proof</u>; The coordinate functions on M are harmonic, of linear growth rate.

$\Rightarrow$ $x^i \in \mathcal{H}_1(M)$.

Apply dimension bound.