Ancient Solutions I

Plan:
- Euclidean case
- Manifold setting

Context

Heat equation: $\partial_t u = \Delta u$

Example ($n=1$): $\partial_t e^{x+t} = e^{x+t} = \partial_x \partial_x e^{x+t}$

so $(x,t) \longmapsto \underline{e^{x+t}}$ is an ancient solution

Souplet–Zhang (Yau's Liouville thm)

$\Rightarrow$ $u \geq 0$, ancient, w/ growth slower than $e^{c\,d(x,x_0) + C\sqrt{t}}$ then it is constant.
(SHARP)

QN. What are the pos. ancient solutions?

Credit: Lin–Zhang '17.

Theorem 1. (Representation Formula)
Let u be a non-neg. soln to the heat equation in $\mathbb{R}^n \times (-\infty, 0]$.

Then $u(x,-t)$ is a completely monotone function in t.

Furthermore $\exists$ a family of non-neg. Borel measures $\mu = \mu(\cdot, s)$ on S^{n-1} & a Borel measure $\rho = \rho(s)$ on $[0,\infty)$ s.t.

$$\boxed{u(x,t) = \int_0^\infty \int_{S^{n-1}} e^{ts + x\cdot\xi\sqrt{s}}\, d\mu(\xi, s)\, d\rho(s)}$$

Remark. Widder '63 u pos. ancient soln in $\mathbb{R}^n \times (-\infty, 0]$ then $u(x,t) = \int_{\mathbb{R}^n} e^{x\cdot y + t|y|^2}\, d\mu(y)$, μ non-neg. Borel measure.

Theorem 2. (Structural Thm) Let $H^q(\mathbb{R}^n \times (-\infty, 0])$ denote the space of anc. solns. sat.

$q \geq 1$

$$|u(x,t)| \leq C\left(d(x,x_0) + \sqrt{|t|} + 1\right)^q, \quad \forall (x,t).$$

Then $\exists C, \eta$ s.t.

$\eta = \log_2 d_0$ $\quad d_0$ doubling dimn

$$\dim\left(H^q(\mathbb{R}^n \times (-\infty, 0])\right) \leq C q^{\eta+1}.$$

linear in n (coeff ≤ 10)

Furthermore, for $k = \lceil \frac{q}{2} \rceil$,

biharmonic

$$u(x,t) = u_0(x) + u_1(x)t + \dots + u_{k-2}(x)t^{k-2} + u_{k-1}(x)t^{k-1}$$

harmonic

where $\Delta u_i(x) = (i+1)u_{i+1}(x)$ ✓ and
$\Delta u_{k-1}(x) = 0.$

Remark. $\Delta^2 u_{k-2} = (k-1)\Delta u_{k-1} = 0$
$\Delta^3 u_{k-3} = 0$ etc.

Proof of theorem 1. Ten steps:

1. Monotonicity (Li-Yau)
2. Bernstein
3. Fubini
4. Inverse Laplace
5. Integral Estimate
6. PDE for $h(\cdot,t)$
7. Harnack
8. Radon-Nikodym
9. PDE for $\frac{d\sigma_x}{d\sigma_0}$
10. Rep formula (Caffarelli-Littman).

We will need

Theorem (Li-Yau) Let u be a pos. soln to the H. Eqn in

$$Q_{R,T}(x,t) = B_R(x) \times [t-T,T].$$

Then $\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \le C_n\left(\frac{1}{R^2} + \frac{1}{T}\right)$ in $Q_{\frac{R}{2},\frac{T}{2}}(x,t)$,

where $C_n = C_n(n)$.

Proof (idea) Set $f = \log u$. Then

$$\partial_t f = \frac{u_t}{u}, \quad \Delta f = \frac{\Delta u}{u} - \frac{|\nabla u|^2}{u^2}$$

$$\Rightarrow \partial_t f = \Delta f + |\nabla f|^2$$

$$\Delta f = \partial_t f - |\nabla f|^2$$

$$\partial_t f = \underline{\Delta f + |\nabla f|^2}$$

$$\Rightarrow \partial_t \Delta f = \Delta\Delta f + \Delta |\nabla f|^2$$
$$= \Delta\Delta f + \operatorname{div}(2\nabla\nabla f \cdot \nabla f)$$
$$= \Delta\Delta f + 2|\nabla^2 f|^2 + 2\nabla\Delta f \cdot \nabla f$$

$$\Rightarrow (\partial_t - 2\nabla_{\nabla f} - \Delta)\Delta f = 2\underline{|\nabla^2 f|^2}$$

Note $|\nabla^2 f|^2 \geqslant \frac{1}{n}|\Delta f|^2$ so

$$(\partial_t - 2\nabla_{\nabla f} - \Delta)\Delta f \geqslant \frac{2}{n}|\Delta f|^2.$$

Also, $(\partial_t - 2\nabla_{\nabla f} - \Delta)\frac{n}{2(t-T)} = \frac{2}{n}\underline{\frac{n^2}{4(t-T)^2}}$

$$\Rightarrow (\partial_t - 2\nabla_{\nabla f} - \Delta)\left(\Delta f + \frac{n}{2(t-T)}\right) \geqslant 0$$

$\Rightarrow$ preservation of $\Delta f \geqslant -\frac{n}{2(t-T)}$

$$\Rightarrow \partial_t f - |\nabla f|^2 \geqslant -\frac{n}{2(t-T)}, \quad f = \log u$$

$$\Rightarrow \frac{u_t}{u} - \frac{|\nabla u|^2}{u^2} \geqslant -\frac{n}{2(t-T)}.$$

For the full result, localise.

Step 1. MONOTONICITY.

For $u \geq 0$ soln to HE in $\mathbb{R}^n \times (-\infty, 0]$,

Li–Yau $\Rightarrow$ $\frac{1}{2}\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \leq C_n\left(\frac{1}{R^2} + \frac{1}{T}\right)$

$\Rightarrow$ $u_t(x,t) \geq -C_n u(x,t)\left(\frac{1}{R^2} + \frac{1}{T}\right)$

$\geq -\varepsilon \quad \forall \varepsilon > 0$

$\Rightarrow$ $u_t \geq 0$.

Step 2. Bernstein.

$u_t \geq 0$, $u_{tt} \geq 0, \ldots, \partial_t^k u \geq 0$.

therefore $(-1)^k \partial_t^k (\underbrace{u(x,-t)}_{f^x(t)}) \geq 0$,

so $(x,t) \longmapsto u(x,-t)$ is a completely monotone function in t, so

(Bernstein) $$f^x(t) = u(x,-t) = \int_0^\infty e^{-ts} \, d\nu(s,x)$$

where $\nu(\cdot,x)$ is a non-neg Borel measure on $[0,\infty)$.

Note that

$$f^x(t) = \underset{\substack{\shortparallel \\ u(x,0)}}{f^x(0)} + \int_0^\infty (e^{-ts}-1) \, d\nu(s,x).$$

Step 3. Fubini

$$f^x(t) = u(x,0) + \int_0^\infty (e^{-ts} - 1)\, d\upsilon(s,x)$$

$$= u(x,0) + \int_0^\infty (-t) \int_0^s e^{-t\lambda}\, d\lambda\, d\upsilon(s,x)$$

FUBINI

$$\int_0^\infty t e^{-t\lambda}\, d\lambda = 1$$

$$= u(x,0) - \int_0^\infty t e^{-t\lambda} \int_\lambda^\infty d\upsilon(s,x)\, d\lambda$$

$$= \int_0^\infty t e^{-t\lambda} \underbrace{\left[u(x,0) - \int_\lambda^\infty d\upsilon(s,x) \right]} d\lambda$$

$$= \int_0^\infty t e^{-ts}\, h(x,s)\, ds .$$

Here $h(x,s) = \int_0^s d\upsilon(s,x)$,

$h(x,\cdot)$ is right cts, non-dec.

Step 4. Inverse Laplace.