Ancient Solutions I

Plan:
- Euclidean case
- Manifold setting

Context

Heat equation: $\partial_t u = \Delta u$

Example: ($n=1$) $\partial_t e^{x+t} = e^{x+t} = \partial_x \partial_x e^{x+t}$

So $(x,t) \longmapsto \underline{e^{x+t}}$ is an ancient solution

Souplet–Zhang (Yau's Liouville thm)

$\Rightarrow$ $u \geq 0$, ancient, w/ growth slower than $e^{c\,d(x,x_0) + C\sqrt{t}}$ then it is constant. (SHARP)

QN. What are the pos. ancient solutions?

Credit: Lin–Zhang '17.

Theorem 1. (Representation Formula)
Let u be a non-neg. soln to the heat equation in $\mathbb{R}^n \times (-\infty, 0]$.

Then $u(x,-t)$ is a completely monotone function in t.

Furthermore $\exists$ a family of non-neg. Borel measures $\mu = \mu(\cdot, s)$ on S^{n-1} & a Borel measure $\rho = \rho(s)$ on $[0,\infty)$ s.t.

$$\boxed{u(x,t) = \int_0^\infty \int_{S^{n-1}} e^{ts + x\cdot\xi\sqrt{s}}\, d\mu(\xi, s)\, d\rho(s)}$$

Remark. Widder '63 u pos. ancient soln in $\mathbb{R}^n \times (-\infty, 0]$ then $u(x,t) = \int_{\mathbb{R}^n} e^{x\cdot y + t|y|^2} d\mu(y)$, μ non-neg. Borel measure.

$q \geq 1$

Theorem 2. (Structural Thm) Let $H^q(\mathbb{R}^n \times (-\infty,0])$ denote the space of anc. solns. s.t.

$$|u(x,t)| \leq C\left(d(x,x_0)+\sqrt{|t|}+1\right)^q, \ \forall (x,t).$$

Then $\exists C, \eta$ s.t. $\eta = \log_2 d_0$ $\quad d_0$ doubling dimn

$$\dim\left(H^q(\mathbb{R}^n \times (-\infty,0])\right) \leq C q^{\eta+1}.$$

linear in η (coeff ≤ 10)

Furthermore, for $k = \lceil \frac{q}{2} \rceil$,

biharmonic

$$u(x,t) = u_0(x) + u_1(x)t + \dots + u_{k-2}(x)t^{k-2} + u_{k-1}(x)t^{k-1}$$

harmonic

where $\Delta u_i(x) = (i+1)u_{i+1}(x)$ ✓ and
$\Delta u_{k-1}(x) = 0$.

Remark. $\Delta^2 u_{k-2} = (k-1)\Delta u_{k-1} = 0$
$\Delta^3 u_{k-3} = 0$ etc.

Proof of theorem 1. Ten steps:

1. Monotonicity (Li-Yau)
2. Bernstein
3. Fubini
4. Inverse Laplace
5. Integral Estimate
6. PDE for $h(\cdot,t)$
7. Harnack
8. Radon-Nikodym
9. PDE for $\frac{d\sigma_x}{d\sigma_0}$
10. Rep formula (Caffarelli-Littman).

We will need

Theorem (Li-Yau) Let u be a pos. soln to the H.Eqn in

$$Q_{R,T}(x,t) = B_R(x)\times[t-T,T].$$

Then
$$\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \le C_n\left(\frac{1}{R^2}+\frac{1}{T}\right) \quad \text{in } Q_{\frac{R}{2},\frac{T}{2}}(x,t),$$
where $C_n = C_n(n)$.

Proof (idea) Set $f = \log u$. Then

$$\partial_t f = \frac{u_t}{u}, \quad \Delta f = \frac{\Delta u}{u} - \frac{|\nabla u|^2}{u^2}$$

$$\Rightarrow \partial_t f = \Delta f + |\nabla f|^2$$

$$\Delta f = \partial_t f - |\nabla f|^2$$

$$\partial_t f = \underline{\Delta f + |\nabla f|^2}$$

$$\Rightarrow \partial_t \Delta f = \Delta\Delta f + \Delta |\nabla f|^2$$
$$= \Delta\Delta f + \operatorname{div}(2\nabla\nabla f \cdot \nabla f)$$
$$= \Delta\Delta f + 2|\nabla^2 f|^2 + 2\nabla\Delta f \cdot \nabla f$$

$$\Rightarrow (\partial_t - 2\nabla_{\nabla f} - \Delta)\Delta f = 2\underline{|\nabla^2 f|^2}$$

Note $|\nabla^2 f|^2 \geq \frac{1}{n}|\Delta f|^2$ so

$$(\partial_t - 2\nabla_{\nabla f} - \Delta)\Delta f \geq \frac{2}{n}|\Delta f|^2.$$

Also, $(\partial_t - 2\nabla_{\nabla f} - \Delta)\dfrac{n}{2(t-T)} = \dfrac{2}{n}\,\underline{\dfrac{n^2}{4(t-T)^2}}$

$$\Rightarrow (\partial_t - 2\nabla_{\nabla f} - \Delta)\left(\Delta f + \frac{n}{2(t-T)}\right) \geq 0$$

$\Rightarrow$ preservation of $\Delta f \geq -\dfrac{n}{2(t-T)}$

$$\Rightarrow \partial_t f - |\nabla f|^2 \geq -\frac{n}{2(t-T)}, \quad f = \log u$$

$$\Rightarrow \frac{u_t}{u} - \frac{|\nabla u|^2}{u^2} \geq -\frac{n}{2(t-T)}.$$

For the full result, localise.

Step 1. MONOTONICITY.

For $u \geq 0$ soln to HE in $\mathbb{R}^n \times (-\infty, 0]$,

Li-Yau $\Rightarrow$ $\frac{1}{2}\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \leq C_n\left(\frac{1}{R^2} + \frac{1}{T}\right)$

$$\Rightarrow \quad u_t(x,t) \geq -C_n u(x,t)\left(\frac{1}{R^2} + \frac{1}{T}\right)$$

$$\geq -\varepsilon \quad \forall\, \varepsilon > 0$$

$$\Rightarrow \quad u_t \geq 0.$$

Step 2. Bernstein.

$u_t \geq 0$, $u_{tt} \geq 0, \dots, \partial_t^k u \geq 0$.

therefore $(-1)^k \partial_t^k (\underbrace{u(x,-t)}_{f^x(t)}) \geq 0$,

so $(x,t) \longmapsto u(x,-t)$ is a completely monotone function in t, so

(Bernstein) $$f^x(t) = u(x,-t) = \int_0^\infty e^{-ts} \, d\nu(s,x)$$

where $\nu(\cdot,x)$ is a non-neg Borel measure on $[0,\infty)$.

Note that

$$f^x(t) = \underbrace{f^x(0)}_{u(x,0)} + \int_0^\infty (e^{-ts}-1) \, d\nu(s,x).$$

Step 3. Fubini

$$\boxed{\int_0^\infty te^{-t\lambda}\,d\lambda = 1}$$

$$f^x(t) = u(x,0) + \int_0^\infty (e^{-ts}-1)\,dv(s,x)$$

$$= u(x,0) + \int_0^\infty (-t)\int_0^s e^{-t\lambda}\,d\lambda\,dv(s,x)$$

Fubini

$$= u(x,0) - \int_0^\infty te^{-t\lambda}\int_\lambda^\infty dv(s,x)\,d\lambda$$

$$= \int_0^\infty te^{-t\lambda}\Big[\underbrace{u(x,0) - \int_\lambda^\infty dv(s,x)}\Big]d\lambda$$

$$= \int_0^\infty te^{-ts}\,h(x,s)\,ds.$$

Here $h(x,s) = \int_0^s dv(s,x)$,

$h(x,\cdot)$ is right cts, non-dec..

Step 4. Inverse Laplace.

So $\displaystyle\int_0^\infty e^{-ts} h(x,s)\,ds = \frac{1}{t} u(x,-t)$

$$\Longrightarrow \quad h(x,t) = \frac{1}{2\pi i}\int_{-1-i\infty}^{-1+i\infty} e^{st}\,\frac{u(x,-s)}{s}\,ds.$$

Step 5. INTEGRAL ESTIMATE.

Recall that $\Delta f^x(t) + \partial_t f^x(t) = 0$ in $\mathbb{R}^4\times[0,\infty)$

$\phi\in C_c^\infty(\mathbb{R}^4)$

$$\Longrightarrow \partial_t \int_{\mathbb{R}^4} f^x(t)\phi(x)\,dx = -\int_{\mathbb{R}^4}\phi(x)\Delta f^x(t)\,dx$$

$$= -\int_{\mathbb{R}^4}\Delta\phi(x)\, f^x(t)\,dx$$

$$= -\int_{\mathbb{R}^4}\int_0^\infty s e^{-ts}\phi(x)\,d\nu(x,s)$$

$$\Longrightarrow \int_{\mathbb{R}^4}\int_0^\infty s e^{-ts}\phi(x)\,d\nu(x,s) = \int_{\mathbb{R}^4}\int_0^\infty e^{-ts}\Delta\phi(x)\,d\nu(x,s)$$

$$\Longrightarrow \int_0^\infty e^{-ts}\int_{\mathbb{R}^4}\left(\Delta\phi(x) - s\phi(x)\right)dx\,d\nu(x,s) = 0$$

$\forall\, t\geq 0,\ \phi\in C_c^\infty(\mathbb{R}^4).$

Step 6. $\boxed{\text{PDE for } h(\cdot,t)}$

Define the signed measure η_ϕ on $[0,\infty)$

by $$d\eta_\phi(s) = \int_{\mathbb{R}^n} (\Delta\phi(x) - s\phi(x))\, d\nu(x,s).$$

$$\Rightarrow \quad LT(\eta_\phi) = 0 \quad \Rightarrow \quad \eta_\phi = 0$$

$$\Rightarrow \quad 0 = \eta_\phi([0,t]) = \int_{\mathbb{R}^n} \Big(\Delta\phi(x) \underbrace{\int_0^t d\nu(x,s)}_{h(x,t)} - \phi(x) \int_0^t s\, d\nu(x,s) \Big) dx$$

$$\boxed{\begin{aligned} &\int_0^t s\, d\nu(x,s) \\ &h(x,u) = \int_0^u d\nu(x,s) \end{aligned}}$$

$$\Rightarrow \quad \int_{\mathbb{R}^n} \Delta\phi(x) \int_0^t d\nu(x,s) = \int_{\mathbb{R}^n} \phi(x) \int_0^t s\, d\nu(x,s)$$

$$= \int_{\mathbb{R}^n} \phi(x) \Big(t\, h(x,t) - \int_0^t h(x,s)\, ds \Big) dx$$

Because $\phi \in C_c^\infty(\mathbb{R}^n)$,

$\Rightarrow$ $h(\cdot,t)$ solves $\Delta h(x,t) = t\, h(x,t) - \int_0^t h(x,s)\, ds$

$\Rightarrow$ h smooth. $\quad (\Delta - t)\, h(x,t) = -\int_0^t h(x,s)\, ds.$

Step 7. (Prep for) Harnack

Since $v_x((a,b]) = h(x,b) - h(x,a)$

we have $\Delta v_x(I) = \Delta h(x,b) - \Delta h(x,a)$

$$= b\,h(x,b) - \int_0^b h(x,s)\,ds - a\,h(x,a) + \int_0^a h(x,s)\,ds$$

$$= b\,h(x,b) - a\,h(x,a) - \int_a^b h(x,s)\,ds$$

$$= b\underbrace{\big(h(x,b) - h(x,a)\big)}_{v_x(I)} - \underbrace{\int_a^b \underbrace{h(x,s) - h(x,a)}_{\geq 0}\,ds}_{\leq 0}$$

$\boxed{(b-a)\,h(x,a) = \int_a^b h(x,a)\,ds \quad h \text{ non-dec}}$

$$\Longrightarrow \quad \Delta v_x(I) \leq b\, v_x(I)$$

$$\Longrightarrow \quad \frac{\Delta v_x(I)}{v_x(I)} \leq b. \qquad (*)$$

From below: $\Delta v_x(I) = b\,h(x,b) - a\,h(x,a) - \int_a^b h(x,s)\,ds$

$$= a\underbrace{\big(h(x,b) - h(x,a)\big)}_{v_x(I)} - \underbrace{\int_a^b \underbrace{h(x,s) - h(x,b)}_{\leq 0}\,ds}_{\geq 0}$$

$$\geq a\, v_x(I)$$

$$\Longrightarrow \quad \frac{\Delta v_x(I)}{v_x(I)} \geq a. \qquad (*)$$

So $c(x) = \dfrac{\Delta v_x(I)}{v_x(I)}$ sat.

$|c(x)| \leq b$ on $\mathbb{R}^n$.

Step 8. Harnack & Radon-Nikodym.

Therefore Harnack for $L = \Delta - c$ implies

$$v_x(I) \leq C(|x|, b)\, v_0(I), \quad I = [0, b].$$

$\Rightarrow$ v_x abs. cts wrt v_0

Radon-Nikodym $\Rightarrow$ $\dfrac{dv_x}{dv_0}(s) \leq C(|x|, b)$, $s \leq b$.

Therefore ---

Step 9. $\boxed{\text{PDE for } \dfrac{dv_x}{dv_0}(s)}$

$$\int_0^\infty e^{-ts} \int_{\mathbb{R}^n} (\Delta\phi(x) - s\phi(x))\, dv_x(s) = 0$$

$$\Rightarrow 0 = \int_0^\infty e^{-ts} \Big(\underbrace{\int_{\mathbb{R}^n} (\Delta\phi(x) - s\phi(x)) \frac{dv_x}{dv_0}(s)\, dx}_{} \Big)\, dv_0(s)$$

This is the LT of "signed meas", so for v_0-a.e. s we have

$$\int_{\mathbb{R}^n} (\Delta\phi(x) - s\phi(x)) \frac{dv_x}{dv_0}(s)\, dx = 0.$$

$$\forall \phi \in C_c^\infty(\mathbb{R}^n)$$

$\frac{d\nu_x}{d\nu_0}(s)$ smooth &

so $$(\Delta - s)\frac{d\nu_x}{d\nu_0}(s) = 0.$$

10. Caffarelli–Littman '82

(c.f. Karpelevič '67, Korányi '79)

show that
$$\frac{d\nu_x}{d\nu_0}(s) = \int_{S^{n-1}} e^{x\cdot\xi\sqrt{s}}\, d\mu_s(\xi)$$
for μ_s a Borel meas. on S^{n-1}.

We are done because
$$u(x,t) = \int_0^\infty e^{st}\, d\nu_x(s)$$
$$\Rightarrow u(x,t) = \int_0^\infty \int_{S^{n-1}} e^{st + x\cdot\xi\sqrt{s}}\, d\mu_s(\xi)\, d\nu_0(s).$$
$\square$

Manifold Setting

$\mathbb{R}^n \longleftrightarrow (M^n, g)$ complete non-neg Ric

$\overset{\text{Li-Yau}}{\Longrightarrow} \quad u(x,t) = \int_0^\infty e^{ts} h(s,x)\, ds$

where h solves

(*) $$(\Delta - s)\, h(s,x) = 0$$

$$x \in M.$$

Key concep: Martin boundary.

Martin ('41) & Murata ('86, '93, '02, '07)

$\Rightarrow \quad h(x) = \int_{\Sigma_0} P_s(x,\omega)\, d\mu(\omega), \quad \mu$

unique non-neg Borel meas. on Σ!

$\mu(\Sigma - \Sigma_0) = 0$

where $\Sigma_0 = \{ [\omega] : P(\cdot, \omega)$ is minimal$\}$

$$P_s(x,y) = \begin{cases} \dfrac{\Gamma_s(x,y)}{\Gamma_s(x_0,y)}, & y \neq x_0 \\ 0, & y = x_0, x \neq y \\ 1, & x = y = x_0 \end{cases}$$

$\forall\, 0 \le v(x) \le P(x,\omega) \Rightarrow v(x) = c\,P(x,\omega)$

Γ_s minimal Green's fn for $(\Delta - s)$.

Theorem [LZ]. Let (M, g) be a complete Riem. mfd w/ non-neg Ric. Suppose u is a non-neg ancient sln to the HE. Then

- $u(x, -t)$ comp. monotone in t
- $\exists$ family of non-neg. Borel measure $\mu = \mu(\cdot, s)$ on the Martin bdy Σ_s' for $(\Delta - s)$, & a Borel ρ on $[0, \infty)$ s.t.

$$u(x,t) = \int_0^\infty \int_{\Sigma_s'} e^{ts} P_s(x, w)\, d\mu(w,s)\, d\rho(s).$$

Questions

- Conditions on (M, g) s.t. $\Sigma_{s_1}' \approx \Sigma_{s_2}'$
- Structure of Σ_s'?
- What is P_s?

Ancient Solutions

Part II. Ancient solutions w./ polynomial growth of order q.

Detour from [LZ]; instead, build on what Ben presented (credit: Ben)

Let u be in A_q (Ancient q) on $M \times (-\infty, 0]$, M non-neg. Ricci curvature.

Thm

$$|u| \leq (1 + d(x, x_0) + \sqrt{|t|})^q \leq C(1+R)^q$$

$$\Rightarrow \left\{ \begin{array}{l} u + C(1+R)^q \\ C(1+R)^q - u \end{array} \right. \text{ non-neg solns to HE}$$

Li-Yau
$$\Rightarrow |u_t| \leq CR^{-2}|u| \leq \tilde{C}R^{q-2}$$

$$\Rightarrow u \in A_q, \text{ then } u_t \in A_{q-2}$$

& if $q < 2$, then $u_t = 0$.

This means that

$$\partial_t^k u = 0 \quad , \quad k = \lceil \tfrac{q}{2} \rceil$$

$\Rightarrow$ u is a polynomial of degree at most $k-1$ in t

$$\Rightarrow \quad u(x,t) = \sum_{i=0}^{k-1} \frac{u_i(x)\, t^i}{i!} . \qquad (1)$$

Differentiating (1) w.r.t. t moves coeffs left,
Differentiating (1) w.r.t. x factors through

$$\Rightarrow \quad \Delta u_{k-1} = 0$$

$$\Rightarrow \quad \Delta \Delta u_{k-2} = \Delta u_{k-1} = 0$$

and so on.

(This is [LZ] Theorem 2.1.)

For the dimension bound, observe that

$$u_{k-1} \in H_q \quad , \quad \dim H_q \leq C(n)\, q^{n-1}$$

$\Delta u_{k-2} = u_{k-1}$, given a part. soln. $\tilde{u}_{k-2}$, the soln space is spanned by $\tilde{u}_{k-2}$ and a harmonic function $\in H_{q-2}$.

This goes all the way down to u_0.

$$\dim A_q \leq C(n)\left(q^{n-1} + (q-2)^{n-1} + (q-4)^{n-1} + \ldots + (q-2k+2)^{n-1}\right)$$
$$\leq \hat{C}(n)\, q^n.$$

That's it!

Theorem. $\dim A_q \leq \hat{C}(n)\, q^n$. For $u \in A_q$,

$$u(x,t) = \sum_{i=0}^{k-1} \frac{u_i(x)}{i!} t^i,$$

where $\Delta u_i = u_{i+1}$, $\Delta u_{k-1} = 0$.