Transcript: Harmonic functions of polynomial growth 2

Ben Andrews (notes taken by Paul Bryan)

October 7 2020

1 Last time

(M^n, g) is a complete, non-compact Riemannian manifold with $\mathrm{Ric} \geq 0$.

1. Laplacian Comparison: $\Delta\rho^2 \leq 2n$, $\rho(x) = d(x, x_0)$
2. Volume comparision: $|B_R(x_0)| \leq \left(\frac{R}{r}\right)^n |B_r(x_0)|$ for $0 < r \leq R$
3. Poincaré inequality: $\int_{B_r(x_0)} |\nabla\varphi|^2 \geq \frac{C(n)}{r^2} \int_{B_r(x_0)} \varphi^2$ for $\varphi|_{B_r(x_0)} = 0$.

2 Yau gradient estimate

Lemma 2.1. *If $u > 0$, $\Delta u = 0$ on $B_r(x_0) \subset M$ then*

$$(r^2 - \rho(x)^2) |\nabla \ln u(x)| \leq c(n)r$$

on $B_r(x_0)$.

Proof. WLOG $r = 1$. We use $\Delta u = 0$, $u > 0$ and let $v = \ln u$.

$$\Delta v = \nabla_i \left(\frac{\nabla_i u}{u}\right) = \frac{\Delta u}{u} - \left|\frac{\nabla u}{u}\right|^2 = -|\nabla v|^2. \tag{1}$$

Let $F = |\nabla v|^2$. Then differentiating (1),

$$0 = \nabla_i(\Delta v + F) = \Delta\nabla_i v - R_i^p \nabla_p u + \nabla_i F$$

Note at a maximum point $|\nabla F| = 2\left|\frac{\nabla\varphi}{\varphi}\right| F$.

Therefore

$$\begin{aligned} 0 &= \Delta F - 2\left|\nabla^2 v\right|^2 - 2\operatorname{Ric}(\nabla v, \nabla v) + 2\nabla v \cdot \nabla F \\ &\leq \Delta F - \frac{2}{n}(\Delta v)^2 + 2\sqrt{F}\,|\nabla F| \\ &= \Delta F - \frac{2}{n}F^2 + 2\sqrt{F}\,|\nabla F| \end{aligned} \tag{2}$$

Write $\varphi = 1 - \rho^2$ with $\rho = d(\cdot, x_0)$. Let $G = \varphi^2 F$. On $\partial B_1(x_0)$, $G = 0$. At x,

$$\begin{aligned} 0 = \nabla G = \varphi^2 \nabla_i F &= 2\varphi\nabla_i\varphi F \\ &= \varphi^2\left(\nabla_i F + \frac{\nabla\varphi}{\varphi^3}G\right) \end{aligned}$$

At x, using (2)

$$\begin{aligned} 0 \geq \Delta G &= \varphi^2(\Delta F + (2\frac{\Delta\varphi}{\varphi^2} - 6\frac{|\nabla\varphi|^2}{\varphi^4})\varphi^2 F) \\ &\geq \varphi^2(\frac{2}{n}F^2 - 2\sqrt{F}2\frac{|\nabla\varphi|}{\varphi}F + (2\frac{\Delta\varphi}{\varphi} - 6\frac{|\nabla\varphi^2|}{\varphi^2}F)) \\ &= F(\frac{2}{n}G - 4\sqrt{G}\,|\varphi| + 2\varphi\Delta\varphi - 6\left|\nabla\varphi^2\right|) \\ &\geq F(\frac{1}{n}G - C\,|\nabla\varphi|^2 + 2\varphi\Delta\varphi) \end{aligned}$$

using Cauchy-Schwarz in the last line.

From $\varphi = 1 - \rho^2$, $\nabla\varphi = -2\rho\,|\nabla\rho| \Rightarrow |\nabla\varphi| \leq 4\rho^2 \leq 4$ and $\Delta\varphi = -\Delta\rho^2 \geq -2n$. Thus

$$G \leq C(n).$$

□

3 Harnack Inequality

Theorem 3.1. *If $u > 0$, $\Delta u = 0$ on $B_r(x_0) \subseteq M$ then*

$$\sup_{B_{r_1}(x_0)} u \leq C(n) \inf B_{r_2}(x_0) u$$

Proof. WLOG $r = 1$. By the gradient estimate, $|\nabla \ln u| \leq C(n)$,

$$\frac{u(y)}{u(x)} \leq e^{C(n)} d(x, y) \leq e^{C(n)}.$$

□

4 Mean Value Inequality (Li-Schoen)

Assume (M, g) is complete, non-compact with $\mathrm{Ric} \geq 0$.

Theorem 4.1. *If v is a non-negative, subharmonic function ($\Delta v \geq 0$) on $B_r(x_0)$, then*

$$v(x_0) \leq c(n) \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} v.$$

Proof in the case $v = u^2$, $\Delta u = 0$.

$$\Delta v = 2 \left|\nabla u\right|^2 \geq 0$$

- **Case 1**: If h is harmonic and $h \geq 0$, the Harnack inequality gives

$$\begin{aligned} h(x_0) &\leq \sup_{B_{r/2}(x_0)} h \leq C(n) \inf_{B_{r/2}} h(x_0) \\ &\leq C(n) \frac{1}{B_{r/2}(x_0)} \int_{B_{r/2}(x_0)} h \\ &\leq C(n) \frac{1}{B_r(x_0)} \int_{B_{r/2}(x_0)} h \frac{B_r(x_0)}{B_{r/2}(x_0)} \\ &\leq 2^n C(n) \frac{1}{B_{r(x_0)}} \int_{B_r(x_0)} h \end{aligned}$$

using volume comparison.

- **Case 2**: $v = u^2$, $\Delta u = 0$. Let h be the harmonic function on $B_{r/2}$ with $h|_{\partial B_r} = |u|$. Note the Harnack gives $h > 0$ on the interior.

Note $|u|$ is subharmonic hence $|u|(x_0) \leq h(x_0)$ so

$$u^2(x_0) \leq h^2(x_0) \leq C(n) \left(\frac{1}{B_{r/2}(x_0)} \int_{B_{r/2}(x_0)} h \right)^2$$
$$\leq C(n) \frac{1}{B_{r/2}(x_0)} \int_{B_{r/2}(x_0)} h^2$$

by Hölder's inequality.

Write

$$\int_{B_{r/2}} h^2 = \int_{B_r} (h - |u| + |u|)^2 \leq 2 \int_{B_{r/2}} (h - |u|)^2 + 2 \int_{B_{r/2}} u^2$$

noting that $h - |u| = 0$ on the boundary. The using the Poincare inequality,

$$\int_{B_{r/2}} (h - |u|)^2 \leq C(n) \int_{B_{r/2}} |\nabla h - \nabla |u||^2 \leq C \int_{B_{r/2}} |\nabla h|^2 + |\nabla u|^2$$

Since h is harmonic, it minimises the Dirichlet energy among maps with the same energy and hence

$$\int_{B_{r/2}} |\nabla h|^2 \leq \int_{B_{r/2}} |\nabla |u||^2 \leq \int_{B_{r/2}} |\nabla u|^2$$

where we note u is smooth hence in $W^{2,2}$.

Let $\Phi \in C_c^\infty(B_1)$, $\Phi \equiv 1$ on $B_{r/2}$, $|\nabla \Phi| \leq C$.

$$\int_{B_1} \Phi |\nabla u|^2 = - \int_{B_1} \Phi^2 u \Delta u - \int_{B_1} 2\Phi u \nabla \Phi \cdot \nabla u$$
$$= 2 \left(\int_{B_1} \Phi^2 |\nabla u|^2 \right)^{1/2} \left(\int_{B_1} u^2 |\nabla \Phi|^2 \right)^{1/2}$$

Thus

$$\int_{B_{1/2}} |\nabla u|^2 \leq \int_{B_1} \Phi^2 |\nabla u|^2 \leq 4 \int_{B_1} u^2 |\nabla \Phi|^2 \leq C \int_{B_1} u^2$$

giving

$$\int_{B_{r/2}} h^2 \leq C \int_{B_1} u^2$$

Using volume comparison,

$$u^2(x_0) \le \frac{C}{|B_{r/2}(x_0)|}\int_{B_1} u^2 \le 2^n C \frac{1}{|B_1|}\int_{B_1} u^2.$$

□

5 Harmonic Functions of Polynomial Growth

Theorem 5.1.

$$\dim \mathcal{H}_p(M) \le C(n)p^{n-1}$$

where

$$\mathcal{H}_p(M) = \{u \in C^\infty(M) : \Delta u = 0, |u(x)| \le C(1 + d(x, x_0)^p\}$$

We aim to bound the dimension of any finite dimensional subspace, K of $\mathcal{H}_p(M)$ which will give the result. First we have an estimate on how harmonic functions can be "packed" into a ball.

Lemma 5.2. *Let K be any finite dimensional subspace of $\mathcal{H}_p(M)$ of $\{u \in C^\infty(M) : \Delta u = 0\}$. Let $\{u_i\}_{i=1}^k$ be any orthonormal basis of K with respect to $L^2(B_r(x))$.*

Then for any $0 < \epsilon < 1/2$,

$$\int_{B_{(1-\epsilon)r}(x)} \sum_{i=1}^k u_i^2 \le C(n)\epsilon^{-(n-1)}.$$

Remark 5.3.

1. The right hand side, $C(n)\epsilon^{-(n-1)}$ is independent of K. Thus while the space of all harmonic functions on a complete, non-compact manifold is infinite dimensional, any finite dimensional space must be concentrated towards the outer edge of the ball. We cannot fit too many harmonic functions into a small space.

2. $\int_{B_r(x)} u_i u_j = \delta_{ij} \Rightarrow \int_{B_r(x)} \sum_{i=1}^k u_i^2 = k$.

Proof. WLOG $r = 1$.

If $y \in B_{1-\epsilon}(x)$ by Gram-Schmidt, choose a rotation, Θ of $\mathbb{R}^k$ such that

$$\Theta \begin{pmatrix} u_1(y) \\ \vdots \\ u_k(y) \end{pmatrix} = \sqrt{\sum_{i=1}^k u_i^2(y)} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Let $u = \sum_i \Theta_i^i u_i$ (harmonic!). The MVI gives

$$u^2(y) \leq C \frac{1}{\left|B_{1-\rho(y)}\right|} \int_{B_{1-\rho(y)}} u^2 \leq C \frac{1}{|B_1(x)|} \int_{B_1(x)} u^2 \frac{|B_1(x)|}{\left|B_{1-\rho(y)}(y)\right|}.$$

Volume comparison doesn't directly apply since we have different centres. But we get

$$u^2(y) \leq C \frac{|B_{1+\rho}(y)|}{|B_{1-\rho}(y)|} \frac{1}{|B_1(x)|} \int_{B_1(x)} u^2 \leq C \left(\frac{1+\rho}{1-\rho}\right)^n \frac{1}{|B_1(x)|} \int_{B_1(x)} u^2.$$

Easy estimate: on $B_{1-\epsilon}$, $\frac{1+\rho}{1-\rho} \leq \frac{2}{\epsilon}$ giving

$$\sum_{i=1}^k u_i^2(y) = u^2(y) \leq \frac{C\epsilon^{-n}}{|B_1(x)|}$$

and hence

$$\int_{B_{1-\epsilon}(x)} \sum_{i=1}^k u_i^2 \leq C\epsilon^{-n} \frac{|B_{1-\epsilon}|}{|B_1|}.$$

More work is required to bump up the estimate to $\epsilon^{-(n-1)}$. □