Transcript: Ancient solutions of the heat equation with polynomial growth 02

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1 Ten step proof of the Representation Formula

1. Monotonicity (via Li-Yau differential Harnack) (proof sketch)
2. Bernstein theorem for completely monotone functions (won't prove)
3. Apply Fubini
4. **Apply inverse Laplace transform**
5. Integral estimate
6. PDE for $h(\cdot, t)$
7. Harnack inequality for measures
8. Radon-Nikodym
9. PDE for $\frac{d\nu_x}{d\nu_0}$
10. Apply representation formula (Caffarelli-Littman) (won't prove)

2 Step 4: Inverse Laplace transform

So

$$\int_0^\infty e^{-ts} h(x,s)ds = \frac{1}{t}u(x,-t).$$

Bernstein gives extension of t into right half complex plane and heat equation estimates give Laplace transform is invertible,

$$h(x,t) = \frac{1}{2\pi i}\int_{-1-i\infty}^{-1+it} e^{st}\frac{u(x,-s)}{s}ds.$$

3 Step 5: Integral estimate

Recall that $\Delta f^x(t) + \partial_t f^x(t) = 0$ in $\mathbb{R}^n \times [0,\infty)$. Testing against a smooth cut-off function $\phi \in C_c^\infty(\mathbb{R}^n \to \mathbb{R})$,

$$\begin{aligned}
\partial_t \int_{\mathbb{R}^n} f^x(t)\phi(x)dx &= -\int_{\mathbb{R}^n} \phi(x)\Delta f^x dx \\
&= -\int_{\mathbb{R}^n} \Delta\phi(x) f^x(t)dx \\
&= -\int_{\mathbb{R}^n}\int_0^\infty se^{-ts}\phi(x)d\nu(x,s).
\end{aligned}$$

Thus

$$\int_{\mathbb{R}^n}\int_0^\infty se^{-ts}\phi(x)d\nu(x,s) = \int_{\mathbb{R}^n}\int_0^\infty e^{-ts}\Delta\phi(x)d\nu(x,s)$$

which is a Laplace transform again:

$$\int_0^\infty e^{-ts}\int_{\mathbb{R}^n}(\Delta\phi(x) - s\phi(x))dxd\nu(x,s) = 0$$

for all $t > 0$ and $\phi \in C_c^\infty(\mathbb{R}^n \to \mathbb{R})$.

4 Step 6: PDE for $h(\cdot,t)$

Define the signed measure η_ϕ on $[0,\infty)$ by

$$d\eta_\phi = \int_{\mathbb{R}^n}(\Delta\phi(x) - s\phi(x))d\nu(x,s).$$

We've shown,

$$\mathrm{LT}(\eta_\phi) = 0 \Rightarrow \eta_\phi = 0.$$

That is,

$$0 = \eta_\phi([0,t]) = \int_{\mathbb{R}^n} \left[\Delta\phi(x) \underbrace{\int_0^t d\nu(x,s)}_{h(x,t)} - \phi(x) \int_0^t s d\nu(x,s) \right] dx.$$

Side note:

$$h(x,u) = \int_0^u d\nu(x,s)$$

and so $h = dx/d\nu(x,s)$ and we can integrate

$$\int_0^t s d\nu(x,s)$$

by parts. Thus

$$\begin{aligned} \int_{\mathbb{R}^n} \Delta\phi(x) \int_0^t d\nu(x,s) &= \int_{\mathbb{R}^n} \phi(x) \int_0^t s d\nu(x,s) \\ &= \int_{\mathbb{R}^n} \phi(x) \left(th(x,t) - \int_0^t h(x,s) ds \right) dx \end{aligned}$$

Because this is true for all $\phi \in C_c^\infty(\mathbb{R}^n \to \mathbb{R})$, h satisfies the integro-differential equation,

$$\Delta h(x,t) = th(x,t) - \int_0^t h(x,s) ds$$

hence h is smooth as the solution of

$$(\Delta - t)\, h(x,t) = -\int_0^t h(x,s) ds.$$

5 Step 7: Prep for Harnack

Fix x and consider the measure ν_x:

$$\nu_x([a,b]) = h(x,b) - h(x,a).$$

Aim is to turn the equation for h into estimates for $\Delta\nu$.

We have (writing $I = [a, b]$),

$$\begin{aligned}
\Delta\nu_x(I) &= \Delta h(x,b) - \Delta h(x,a) \\
&= bh(x,b) - \int_0^b h(x,s)ds \\
&\quad - ah(x,a) + \int_0^a h(x,s)ds \\
&= bh(x,b) - ah(x,a) - \int_a^b h(x,s)ds \\
&= b(\underbrace{h(x,b) - h(x,a)}_{\nu_x(I)}) - \int_a^b \underbrace{h(x,s) - h(x,a)}_{\geq 0} ds
\end{aligned}$$

using side calculation: $(b-a)h(x,a) = \int_a^b h(x,a)ds$ and recalling h is non-decreasing.

Thus,

$$\Delta\nu_x(I) \leq b\nu_x(I)$$

or equivalently,

$$\frac{\Delta\nu_x(I)}{\nu_x(I)} \leq b.$$

Similarly, from below,

$$\begin{aligned}
\Delta\nu_x(I) &= bh(x,b) - ah(x,a) - \int_a^b h(x,s)ds \\
&= a(\underbrace{h(x,b) - h(x,a)}_{\nu_x(I)}) - \int_a^b \underbrace{h(x,s) - h(x,b)}_{\leq 0} ds \\
&\geq a\nu_x(I).
\end{aligned}$$

Thus

$$\frac{\Delta\nu_x(I)}{\nu_x(I)} \geq a.$$

Let

$$c(x) = \frac{\Delta\nu_x(I)}{\nu_x(I)}$$

satisfies,

$$|c(x)| \leq b$$

6 Step 8: Harnack

The Harnack inequality for $L = \Delta - C$ with $I = [0, b]$ gives,

$$\nu_x(I) \le c(|x|, b)\nu_0(I).$$

That is ν_x is absolutely continuous with respect to ν_0.

Radon-Nikodym then gives,

$$\frac{d\nu_x}{d\nu_0}(s) \le c(|x|, b), \quad s \le b.$$

7 Step 9: PDE for $\frac{d\nu_x}{d\nu_0}$

From the integral estimate we obtained,

$$\int_0^\infty e^{-ts} \int_{\mathbb{R}^n} (\Delta\phi(x) - s\phi(x)) d\nu_x(s) = 0.$$

Now we are able to make a change of variables:

$$0 = \int_0^\infty e^{-ts} \left(\int_{\mathbb{R}^n} (\Delta\phi(x) - s\phi(x)) \frac{d\nu_x}{d\nu_0}(s) dx \right) d\nu_0(s)$$

This is the Laplace transform of the signed measure $\int_{\mathbb{R}^n} (\Delta\phi(x) - s\phi(x)) \frac{d\nu_x}{d\nu_0}(s) dx$ so for ν_0-a.e. s we have,

$$\int_{\mathbb{R}^n} (\Delta\phi(x) - s\phi(x)) \frac{d\nu_x}{d\nu_0}(s) dx = 0$$

for every $\phi \in C_c^\infty(\mathbb{R}^n \to \mathbb{R})$. So, for ν_0-a.e. s,

$$(\Delta - s)\frac{d\nu_x}{d\nu_0} = 0$$

and $\frac{d\nu_x}{d\nu_0}$ is smooth.

8 Step 10: Caffarelli-Littman ‘82

See also Kaprelovic ‘67 and Koranyi ‘79.

From

$$(\Delta - s)\frac{d\nu_x}{d\nu_0} = 0$$

we have the representation formula,

$$\frac{d\nu_x}{d\nu_0}(s) = \int_{\mathbb{S}^{n-1}} e^{x\cdot\xi\sqrt{s}} d\mu_s(\xi)$$

for μ_s a Borel measure on $\mathbb{S}^{n-1}$.

This completes the proof because,

$$u(x, -t) = \int_0^\infty e^{-st} d\nu_x(s)$$

implies

$$u(x, t) = \int_0^\infty \int_{\mathbb{S}^{n-1}} e^{st + x\cdot\xi\sqrt{s}} d\mu_s(\xi) d\nu_0(s).$$

$\square$

9 Remarks on Manifold Setting

Replace $\mathbb{R}^n$ with a Riemannian manifold (M^n, g).

For (M, g) complete with Ric ≥ 0 Li-Yau still applies to give

$$u(x, t) = \int_0^\infty e^{ts} h(s, x) dx$$

where h solves

$$(\Delta - s)h(s, x) = 0.$$

Classifying solutions of this equation is the hard part.

Key concept: Martin boundary. (Martin ’41, Murata (’86, ’93, ’02, ’07)

The general theory gives,

$$h(x) = \int_{\Sigma_0} P_s(x, w) d\mu(w)$$

for μ a unique, non-negative Borel measure on Σ with $\mu(\Sigma - \Sigma_0) = 0$.

Here,

$$\Sigma_0 = \{[w] : P(\cdot, w) \text{ is minimal}\}$$

where the term *minimal* means that if $0 \leq v(x) \leq P(x,w)$ then $v(x) = P(x,w)$. Also,

$$P_s(x,y) = \begin{cases} \frac{\Gamma_s(x,y)}{\Gamma_s(x_0,y)}, & y \neq x_0 \\ 0, & y = x_0, x \neq y \\ 1, & x = y = x_0. \end{cases}$$

Γ_s is the minimal Green's function for $\Delta - s$.

Theorem 9.1 (Lin-Zhang)**.** *Let (M,g) be a complete Riemannian manifold with* Ric ≥ 0*. Suppose u is a non-negative, ancient solution of $\partial_t u = \Delta_g u$. Then*

- *$u(x,-t)$ is completely monotone in t,*
- *$\exists$ family of non-negative Borel measures $\mu = \mu(\cdot, s)$ on the Martin boundary, Σ_s for $\Delta - s$, and a Borel measure ρ on $[0,\infty)$ such that*

$$u(x,t) = \int_0^\infty \int_{\Sigma_s} e^{ts} P_s(x,w) d\mu(w,s) d\rho(s).$$

Remark 9.2. The condition Ric ≥ 0 is not strictly necessary. It may be replaced by something like volume doubling and Poincare inequality.

10 Questions

- What are conditions on (M,g) such that

  $$\Sigma_{s_1} \simeq \Sigma_{s_2}$$

  in some sense such as under the motion of a Killing or conformal Killing field.

- Are there conditions determining the structure of Σ_s (compare Schoen for bounded sectional curvature).

- What is P_s?