Transcript: A family of 3d steady gradient solitons that are flying wings

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1 Steady Gradient Solitons (SGS)

Definition 1.1. A steady gradient soliton is a metric g such that

$$\mathrm{Ric} = \frac{1}{2}\mathcal{L}_V\, g = \nabla^2 f.$$

where $V = \nabla f$.

Given such a metric, let

$$g(t) = \varphi_t^* g$$

where φ_t is the flow of $-\nabla f$ and $t \in (-\infty, \infty)$. Then

$$\partial_t g = \varphi_t^*\, \mathcal{L}_{-\nabla f}\, g = -2\varphi_t^* \mathrm{Ric}(g) = -2\,\mathrm{Ric}(g(t))$$

so g is an eternal RF (Ricci Flow).

Example 1.2. Hamilton's Cigar Solition Σ^2:

$$g = dr^2 + \varphi^2(r) d\theta^2$$

with $\theta \in [0, 2\pi)$ and where $\lim_{r\to\infty} \varphi(r) < \infty$.

$$(\Sigma, p_i) \to \mathbb{R} \times \mathbb{S}^1$$

Σ collapsed.

Example 1.3. Bryan solition: $n \geq 3$.

$$g = dr^2 + \varphi^2(r) g_{\mathbb{S}^{n-1}}$$

with

$$\text{warping asymptotics: } \varphi(r) \simeq r^{1/2}$$

$$\text{Scalar curvature asymptotics: } R \simeq r^{-1}$$

$$(\Sigma, R(p_i)g, p_i) \to \mathbb{R} \times \mathbb{S}^{n-1}$$

non-collapsed.

2 Some classification of 3d SGS

Known 3d SGS (non-flat): $\mathbb{R} \times$ Cigar (gradients) and Bryan solition $n = 3$.

1. $\mathrm{Rm} \not> 0$: By the maximum principle, $\mathbb{R} \times$ Cigar.
2. (Brendle) non-collapsed implies Bryant soliton
3. (Deng-Zhu) $\frac{C^{-1}}{r} \leq R \leq \frac{C}{r}$ implies Bryant soliton.
4. (Catino-Monticelli-Mastrolia) If $\lim_{r\to\infty} \int_{B(p,r)} \frac{1}{r} R = 0$ implies quotients of $\mathbb{R} \times$ Cigar.

3 Hamilton's Flying Wing Conjecture

Hamilton's conjecture: there exists a 3d SGS that is a flying wing.

Definition 3.1. A flying wing is a SGS (M, g) with $\mathrm{Rm} \geq 0$ such that the blow-down is $C_\infty(g) = \mathrm{Cone}([0, \theta])$, $\theta \in (0, \pi)$.

The previous examples are not flying wings: $\mathbb{R} \times$Cigar has $\theta = \pi$ while the Bryant solition has $\theta = 0$. Flying wings also do not fit into the classification results above.

4 Flying Wings in MCF

- Constructed by X.J Wang in $\mathbb{R}^n$, $n \geq 3$
- Complete existence results by Hoffman-Ilmanen-Martin-White and Bourni-Langford-Tingalia
- Uniquness by HIMW in $n = 3$ and by BLT in $n \geq 3$.

5 Analogies

	MCF	RF
$n = 2$	Grim Reaper	Cigar Soliton
$n = 3$	Collapsed: Flying Wings New examples (BLT). Non collapsed: Bowl soliton	Collapsed: Flying wing (Thm 1, ($n = 3$) and Thm 2) Non collaped: Bryant soliton
$n \geq 4$	Collapsed: Flying wings. non-collapsed:	Collapsed: Flying wings? Non collapsed: Thm 1 ($n \geq 4$)

6 Theorems

Theorem 6.1 (Thm 1)**.** *$\forall \alpha \in (0, 1)$ $\exists$ a $\mathbb{Z}_2 \times O(n-1)$-symmetric SGS (M, g, f, p) with $\mathrm{Rm} > 0$ such that*

$$\lambda_1 = \alpha\lambda_2 = \cdots = \alpha\lambda_n$$

where λ_i are eigenvalues of Ric at p.

Proof. $n = 3$

Let $\{X_{iu}, u \in [0, 1]\}_{i=1}^{\infty}$ be a sequence of smooth families of metrics on $\mathbb{S}^2$ such that

1. $\mathbb{Z}_2 \times O(2)$-symmetric
2. $K(X_{iu}) > 1$
3. $X_{i0} = c_i g_{\mathbb{S}^2}$, $c_i > 0$,
4. diam($X_{i1} \to \pi$ as $i \to \infty$,

5. $\sup_{u\in[0,1]} \mathrm{vol}(X_{iu}) \to 0$ as $i \to \infty$.

Recall Dernelle's result:

There exists a unique EGS (Expanding Gradient Soliton) (M_{in}, g_{in}, p_{in}) with $\mathrm{Rm} > 0$ such that $C_\infty(g_{in}) = C(X_{in})$ and

- $R(p_{in}) = 1$
- $X_{in} \to X_{in} \overset{\text{Dernelle}}{\to} (M_{in}, g_{in}, p_{in})$

Lemma 6.2. *A sequence of EGS with $\mathbb{Z}_2 \times O(n-1)$-symmetry and $\mathrm{Rm} > 0$ with $R(p_i) = 1$ and $AVR(g_i) \to 0$ as $i \to \infty$ then*

$$(M_i, g_i, p_i) \overset{Cheeger-Gromov}{\to} (M, g, p) \quad SGS\ R(p) = 1.$$

Using Dernelle's result and the lemma

$$(M_{i0}, g_{i0}, p_{i0}) \to \text{Bryant soliton (by rotational symmetry)}, \quad \frac{\lambda_1}{\lambda_3} = 1.$$

and

$$(M_{i1}, g_{i1}, p_{i1}) \to \mathbb{R} \times \text{Cigar}, \quad \frac{\lambda_1}{\lambda_3} = 0.$$

For any $\alpha_0 \in (0,1)$ there exists $u_i \in (0,1)$ such that $\frac{\lambda_1}{\lambda_3}(g_{iu_i}) = 0$. Applying the lemma

$$(M_{in_i}, g_{in_i}, p_{in_i}) \to (M, g, p) \text{ a SGS} \quad \frac{\lambda_1}{\lambda_3}(g) = \alpha_0 \text{ at } p$$

□

The next theorem excludes the blow down is a ray so the construction of Thm 1 gives flying wings.

Theorem 6.3 (Thm 2)**.** *Let (M, g, p) be a $\mathbb{Z}_2 \times O(2)$-symmetric 3d SGS, and $C_\infty(g) = $ a ray $= \mathrm{Cone}(\{0\})$. Then it is a Bryant soliton.*

Proof. In order to obtain a contradiction, suppose it is not a Bryant soliton.

The profile curve Γ is fixed by the $O(2)$ action and the section Σ is fixed by the $\mathbb{Z}_2$ action. Take a geodesic in Σ with $\gamma(0) = p$. Away from the edges

$$g = g_0 + \varphi^2 d\theta, \theta \in [0, \pi)$$

with g_0 totally geodesic.

Define

- $h_1(s) = d(\gamma(s), \Gamma)$
- $h_2(s) = \varphi(\gamma(s))$

Computing gives

$$\frac{h_2'(2)}{h_2(s)} \leq CR(\gamma(s))$$

Lemma 6.4.

$$R(\gamma(s)) \leq Ch_1^{-2}(s)$$

Lemma 6.5.

$$\frac{h_2(s)}{h_1(s)} \to 0, s \to \infty$$

Lemma 6.6.

$$h_1(s)h_2(s) \geq Cs, \quad s \to \infty$$

Using the lemmas,

$$h_2(s) \leq Cs^\epsilon \forall \epsilon < 1/2$$

and

$$h_2(s) \leq C \to \lim_{s\to\infty} R(\Gamma(s)) > 0$$

Lemma 6.7. *Let (M, g) be a $\mathbb{Z}_2 \times O(2)$-symmetric SGS. If*

$$C_\infty(g) = \mathrm{Cone}([0, \alpha]), \alpha \in [0, \pi]$$

Then

$$\lim_{s\to\infty} R(\Gamma(s)) = R(p) \sin \tfrac{\alpha}{2}$$

We have

$$\begin{aligned} 0 \simeq \langle \nabla f, \sigma'(\gamma) \rangle |_{-D}^{D} &= \int_{-D}^{D} \partial_r \langle \nabla f, \sigma'(\gamma) \rangle \, dr \\ &= \int_{-D}^{D} \mathrm{Ric}(\sigma'(\gamma), \sigma'(\gamma)) dr \\ &= 2R^{1/2}(\Gamma(s)) \end{aligned}$$

Thus

$$\lim_{s\to\infty} R(\Gamma(s)) = 0$$

contradicting $\mathrm{Rm} > 0$ for the case $\alpha = 0$. □