MATH704 DG Sem 2, 2018: Assignment 02

Instructions: - Due 10th October

Your grade will be determined from your 3 best answers. Each question has three parts worth 5 points each giving a maximum of 45 points in total. Feel free to turn in answers for all four questions, but only the three best will count.

1 Question 01: Geometry of surfaces of revolution

Let $f > 0$ be a smooth positive function. A surface of revolution is a set $S \subseteq \mathbb{R}^3$ of the form

$$S = \{(x, f(x)\cos\theta, f(x)\sin\theta) : a < x < b, 0 \leq \theta \leq 2\pi\}$$

1. Determine local parametrisations for S. *Hint*: Note carefully that we have $0 \leq \theta \leq 2\pi$ and think about parametrising $\mathbb{S}^1$.
2. Let φ be the local parametrisation defined on $(a, b) \times (0, 2\pi)$. Calculate the coordinate tangent vectors $\partial_x\varphi$ and $\partial_\theta\varphi$ in terms of the function f.
3. Compute the coefficients of the metric g in the parametrisation above.

2 Question 02: Smooth functions on surfaces

For this question, you may assume that given any $p \in S$, there is a local parametrisation $\varphi : U \subseteq \mathbb{R}^2 \to S$ with $p \in \varphi(U)$ and a smooth extension $\Phi : U \times (-\epsilon, \epsilon) \to \mathbb{R}^3$ such that

- $\Phi(u, v, 0) = \varphi(u, v)$,
- $\Phi^{-1} : W \to U \times (-\epsilon, \epsilon)$ is defined and smooth on the open set $W = \Phi(U \times (-\epsilon, \epsilon)) \subseteq \mathbb{R}^3$.

1. Show that if $F : \mathbb{R}^3 \to \mathbb{R}$ is a smooth map and $S \subseteq \mathbb{R}^3$ is a regular surface, then $F|_S : S \to \mathbb{R}$ is a smooth map.
2. Show that for a regular surface S if $f : S \to \mathbb{R}$ is a smooth map, then for every point $p \in S$, there exists an open set $W \subseteq \mathbb{R}^3$ with $p \in S$ and a smooth function $F : W \to \mathbb{R}$ such that $F|_{W\cap S} = f|_{W\cap S}$.
3. Show that a curve $\gamma : (a, b) \to S$ is smooth if and only if it is smooth as a map $(a, b) \to \mathbb{R}^3$.

3 Question 03: Local graphs

1. Let S be a regular surface. Show that locally S is a graph in the sense that for any $x \in S$, there is an open set $U \subset \mathbb{R}^3$ containing x such that

$$S \cap U = \{(x, y, f(x, y))\} \quad \text{or} \quad \{(x, f(x, z), z)\} \quad \text{or} \quad \{(f(y, z), y, z)\}$$

   for a smooth function f of two variables defined on an open set of $\mathbb{R}^2$.

   Hint: Write $\varphi(u, v) = (\varphi_1(u, v), \varphi_2(u, v), \varphi_3(u, v))$ for a local parametrisation. Since the differential $d\varphi$ is injective, show that at least one of $\pi_{xy} \circ \varphi = (\varphi_1, \varphi_2)$, $\pi_{xz} \circ \varphi = (\varphi_1, \varphi_3)$ or $\pi_{yz} \circ \varphi = (\varphi_2, \varphi_3)$ has invertible differential hence we may apply the inverse function theorem to write (u, v) as a smooth function of two out of the three variables (x, y, z). For example when $\pi_{xy} \circ \varphi$ has invertible derivative, (u, v) can be written as a function of (x, y). Conclude that in this case z is a function of (x, y).

2. Consider the *one-sheeted cone*:

$$C = \{z^2 = x^2 + y^2 : z \geq 0\} \subseteq \mathbb{R}^3.$$

   Show that the parametrisation, $\varphi(u, v) = (u, v, \sqrt{u^2 + v^2})$ is not smooth at $(0, 0)$, so certainly cannot be regular.

3. Show that in fact, there are no possible smooth, regular parametrisations.

   Hint: First show that C cannot be written as a graph over the xz or the yz planes in a neighbourhood of $(0, 0, 0)$. Then by part 1, if C is a regular surface, in a neighbourhood of $(0, 0, 0)$, it can be written as the graph of a smooth function: $C = (x, y, f(y, z))$. Show this contradicts part 2.

4 Question 04: Archimedes' cylinder to sphere map

Let $C = \{x^2 + y^2 = 1, -1 < z < 1\}$ denote the unit cylinder and $\mathbb{S}^2 = \{x^2 + y^2 + z^2 = 1\}$ the unit sphere. Define the map

$$\pi(x, y, z) = (\sqrt{1 - z^2}x, \sqrt{1 - z^2}y, z)$$

from $\mathbb{R}^2 \to \mathbb{R}^2$.

1. Show that $\pi|_C$ maps to $\mathbb{S}^2$. Working in *cylindrical polar coordinates*:

$$\varphi_C(r, \theta) = (\cos\theta, \sin\theta, r), \quad \varphi_{\mathbb{S}^2}(r, \theta) = (\sqrt{1 - r^2}\cos\theta, \sqrt{1 - r^2}\sin\theta, r)$$

   on the sphere and cylinder respectively, write down an expression for the map $\varphi_{\mathbb{S}^2}^{-1} \circ \pi \circ \varphi_C$. Using this expression show that π is a smooth map from $C \to \mathbb{S}^2$. Add a line or two about why symmetry implies we only need work in this coordinate expression.

2. Calculate the coordinate tangent vectors for the cylindrical parametrisations of both the cylinder and sphere.

3. Working still in cylindrical polar coordinates, show that the metrics g_C and $g_{\mathbb{S}^2}$ differ. *As an aside, this is not enough to conclude the geometry differs since the difference may be caused by differences in coordinates.* Show also that in these coordinates $\sqrt{\det g_C} = \sqrt{\det g_{\mathbb{S}^2}}$ and hence π is area preserving.