MATH704 DG Sem 2, 2018: Assignment 04

1 Question 01: Immersions and Submersions

1. Let $F : \mathbb{R}^n \to \mathbb{R}^{n+k}$ be a smooth map such that dF_x has rank n for every $x \in \mathbb{R}^n$. Such a map is called an *immersion*. Prove that for each $x \in \mathbb{R}^n$, there is an open set $U \subseteq \mathbb{R}^n$ with $x \in U$ and an open set W with $F(x) \in W$ and local charts $\varphi : U \to \mathbb{R}^n$, $\psi : W \to \mathbb{R}^{n+k}$ such that
$$\psi \circ F \circ \varphi^{-1}(x^1, \ldots, x^n) = (x^1, \ldots, x^n, \underbrace{0, \ldots, 0}_{k \text{ times}}).$$
[5]

2. Let $F : \mathbb{R}^{n+k} \to \mathbb{R}^n$ be a smooth map such that dF_x has rank n for every $x \in \mathbb{R}^n$. Such a map is called a *submersion*. Prove that for each $x \in \mathbb{R}^{n+k}$, there is an open set $U \subseteq \mathbb{R}^{n+k}$ with $x \in U$ and an open set W with $F(x) \in W$ and local charts $\varphi : U \to \mathbb{R}^{n+k}$, $\psi : W \to \mathbb{R}^n$ such that
$$\psi \circ F \circ \varphi^{-1}(x^1, \ldots, x^n, x^{n+1}, \ldots, x^{n+k}) = (x^1, \ldots, x^n).$$
[5]

2 Question 02: Frenet-Serret Equations on an Oriented Surface

Let S be an oriented, regular surface with unit normal ν, metric g, and covariant derivative ∇. Let $\gamma : (a, b) \to S$ be a *regular curve*. That is $\gamma'(t) \neq 0$ for all $t \in (a, b)$. Let $T = \frac{\gamma'}{|\gamma'|_g}$ be the unit tangent vector to γ.

1. Show that $N = \nu \times T$ is tangent to S and g orthogonal to T. Thus N is the intrinsic, oriented unit normal to γ.
[2]

2. Show that $\{T, N, \nu\}$ is positively oriented - that is, it satisfies the right hand rule, $\nu = T \times N$.
[2]

3. Using metric compatibility and the fact that $|T|_g = 1$ show that
$$g(\nabla_T T, T) = 0.$$
[2]

4. Define the *geodesic curvature* $\kappa = g(\nabla_T T, N)$. Verify the Frenet-Serret equations

$$\begin{pmatrix} \nabla_T T \\ \nabla_T N \end{pmatrix} = \begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix} \begin{pmatrix} T \\ N \end{pmatrix}.$$

[4]

5. Let $\kappa_{\mathbb{R}^3}$ be the curvature of γ though of as a curve in $\mathbb{R}^3$ and let $\kappa_\nu = A(T,T)$ be the normal curvature of γ. Show that

$$\kappa_{\mathbb{R}^3}^2 = \kappa^2 + \kappa_\nu^2.$$

Hint: Use the definition of ∇: $\nabla_X Y = D_X Y - \langle D_X Y, \nu \rangle \nu$.

[5]

3 Question 03: Geodesics

Retain the notation of the Frenet-Serret question for $S, \nu, \gamma, T, N, \kappa$. A geodesic is a curve with zero acceleration $\nabla_{\gamma'}\gamma' \equiv 0$. A vector field Y is *parallel* along γ if $\nabla_{\gamma'} Y = 0$. Thus a geodesic is a curve for which the velocity vector is parallel.

1. Using metric compatibility, show that if Y is a parallel vector field along a geodesic γ, then

$$g(\gamma', Y) \text{ and } |Y|_g$$

are constant.

[2]

Conclude that geodesics always have constant speed $|\gamma'|_g$.

[2]

2. Prove that γ is a geodesic if and only if $D_T T$ is normal to S if and only if $\kappa \equiv 0$.

[2]

3. Let $\gamma = P \cap \mathbb{S}^2$ where P is a plane in $\mathbb{R}^3$ through the origin. Prove that γ is a geodesic.

[2]

4 Question 04: Regular Values

Let $F : \mathbb{R}^{n+1} \to \mathbb{R}$ be a smooth function.

1. State the definition of *regular value* $y \in \mathbb{R}$.

[2]

2. Let $y \in \mathbb{R}$ be a regular value. Show that

$$M = F^{-1}(y) := \{x \in \mathbb{R}^{n+1} : F(x) = y\}$$

is a regular hyper-surface.

[5]

3. Show that the unit normal to $M = F^{-1}(y)$ is $N = \frac{\nabla F}{|\nabla F|}$ where ∇F is the usual (Euclidean) gradient of F. *Be sure to show N is well defined!*

[3]

4. Let $\varphi : M \to \mathbb{R}$ be a smooth function. Recall locally we may extend φ to $\Phi : U \to \mathbb{R}$ where $U \subseteq \mathbb{R}^{n+1}$ is an open set. Define the *intrinsic gradient*, $\nabla \varphi$ of φ by

$$\nabla \varphi(x) = \pi_{TM}(\nabla_{\mathbb{R}^{n+1}} \Phi(x)) := \nabla_{\mathbb{R}^{n+1}} \Phi(x) - \langle \nabla_{\mathbb{R}^{n+1}} \Phi(x), N(x) \rangle \, N(x), \quad x \in M.$$

where $\nabla_{\mathbb{R}^{n+1}}$ denotes the Euclidean gradient.

Show that $\nabla \varphi$ is well defined independently of the choice of extension Φ.

Hint: Recall that any vector $Z \in \mathbb{R}^{n+1}$ is uniquely determined by $\langle Z, E_i \rangle$ where E_i is a basis. Then since $\nabla \varphi(x)$ is orthogonal to $N(x)$, it is uniquely determined by $\langle \nabla \varphi(x), V \rangle$ where V ranges over T_xM. Now use $\langle \nabla_{\mathbb{R}^{n+1}} \Phi, V \rangle = d\Phi(V)$ for any $V \in \mathbb{R}^{n+1}$ along with the facts that for $V \in T_xM$, $d\Phi(V)$ may be computed as the derivative along a curve γ on M and that all extensions Φ, Ψ satisfy $\Phi|_M = \Psi|_M$.

[5]

5. Determine a function F and a $y \in \mathbb{R}$ such that y is a regular value and $\mathbb{S}^n = F^{-1}(y)$. Verify that the normal computed as the gradient above is $N(x) = \pm x$ for $x \in \mathbb{S}^n$.

[2]