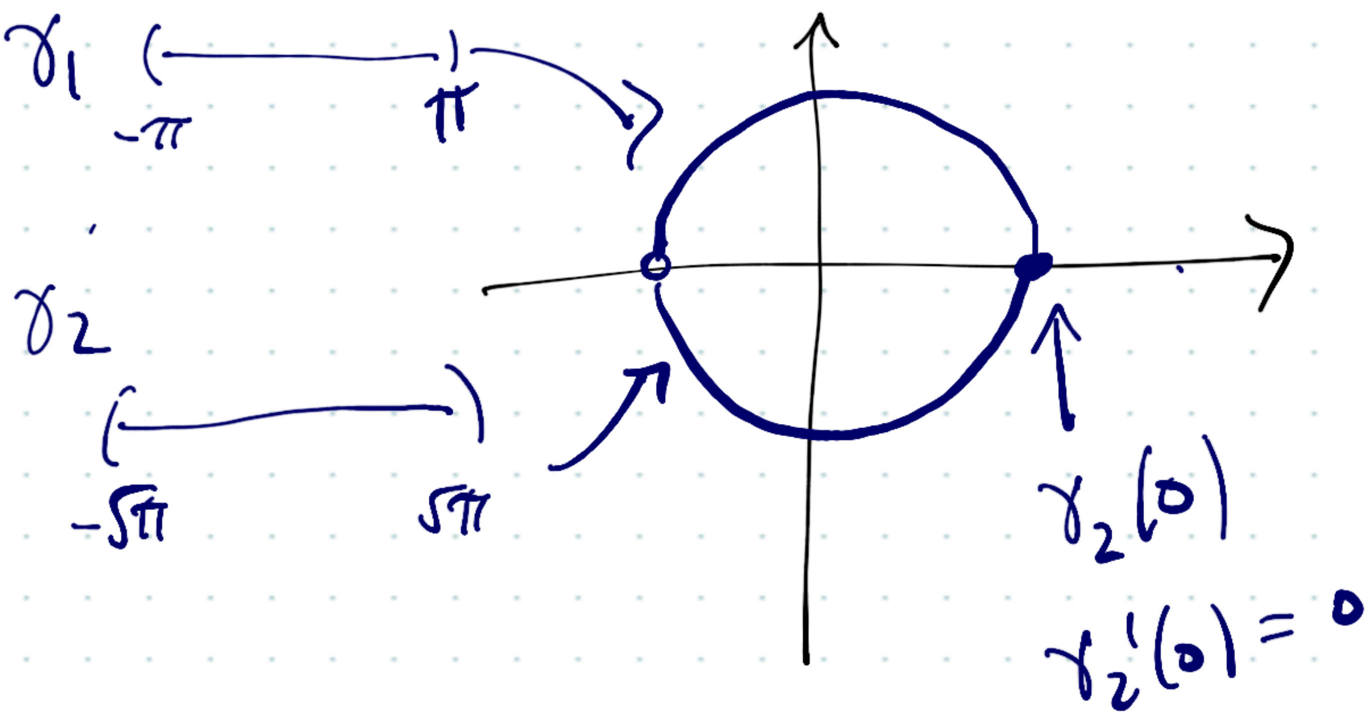
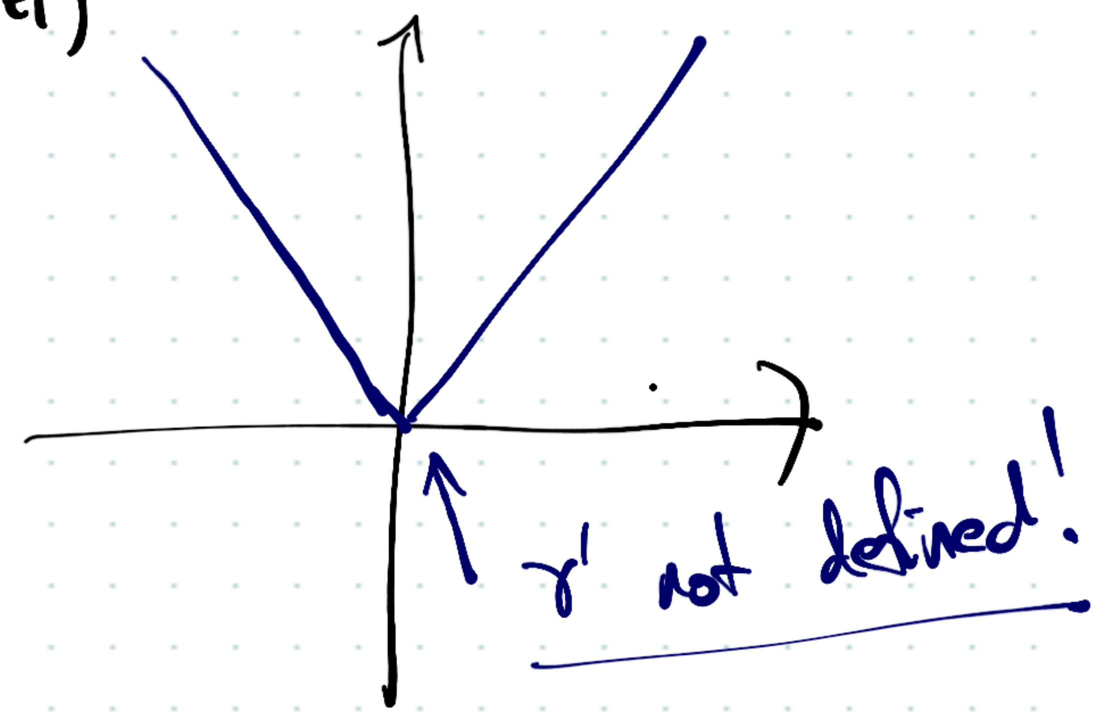
PARAMETRISED CURVES

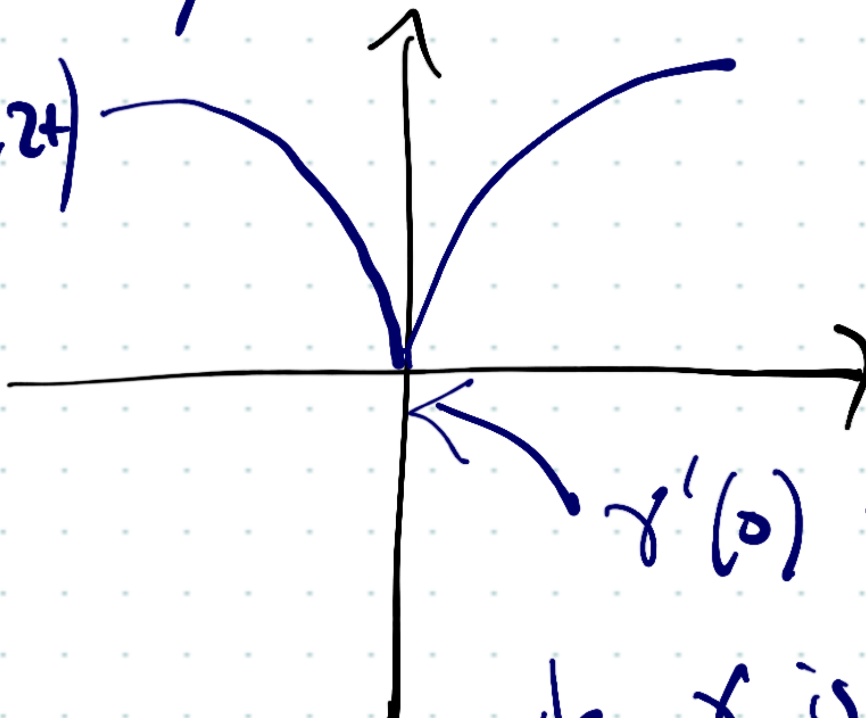


$\gamma(t) = (t, |t|)$



$$\gamma(t) = (t^3, t^2)$$

$$\gamma'(t) = (3t^2, 2t)$$

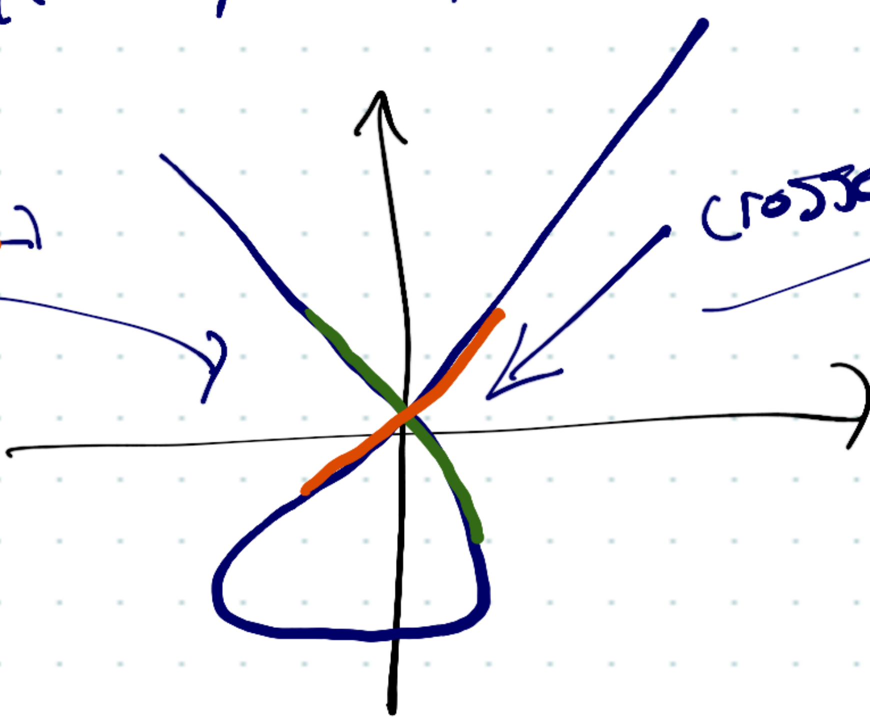


$$\gamma'(0) = 0$$

note γ is smooth!

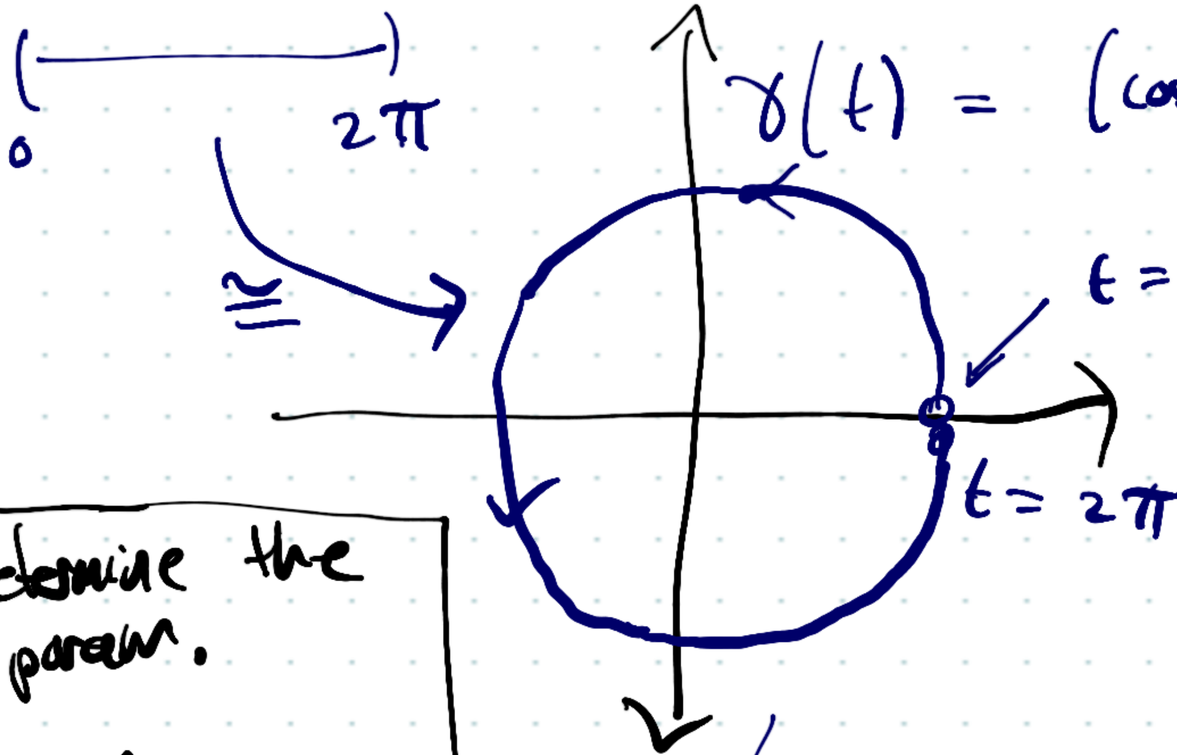
$$\gamma(t) = (t^3 - 4t, t^2 - 4)$$

separated!

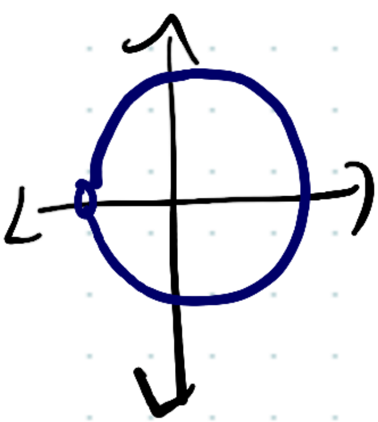


crosses itself!

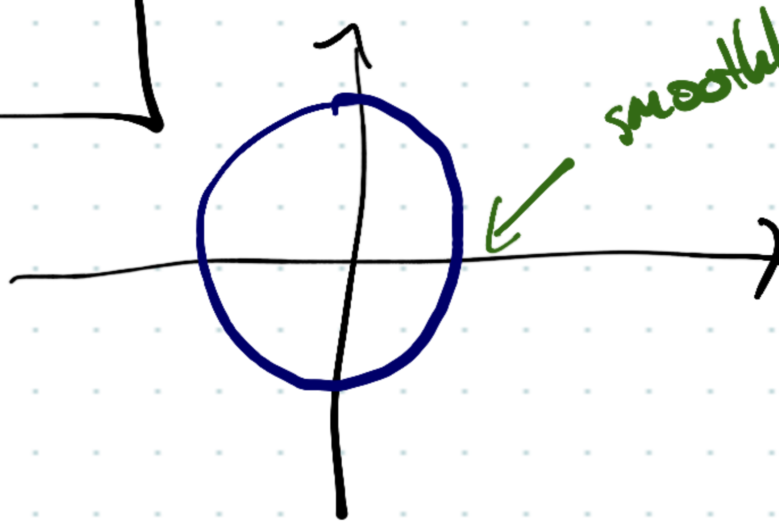
$$\gamma(t) = (\cos(t), \sin(t))$$



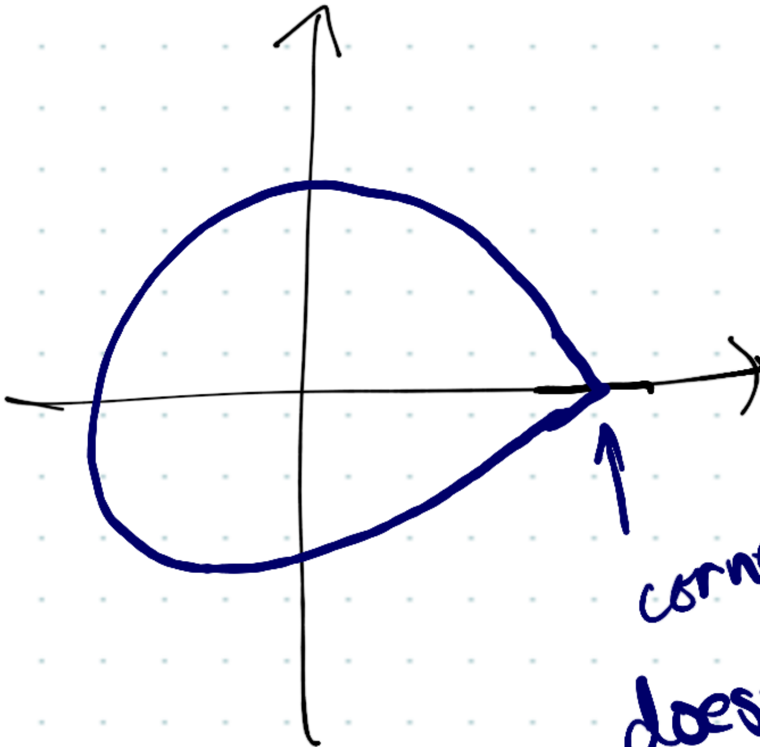
determine the param.



closes up

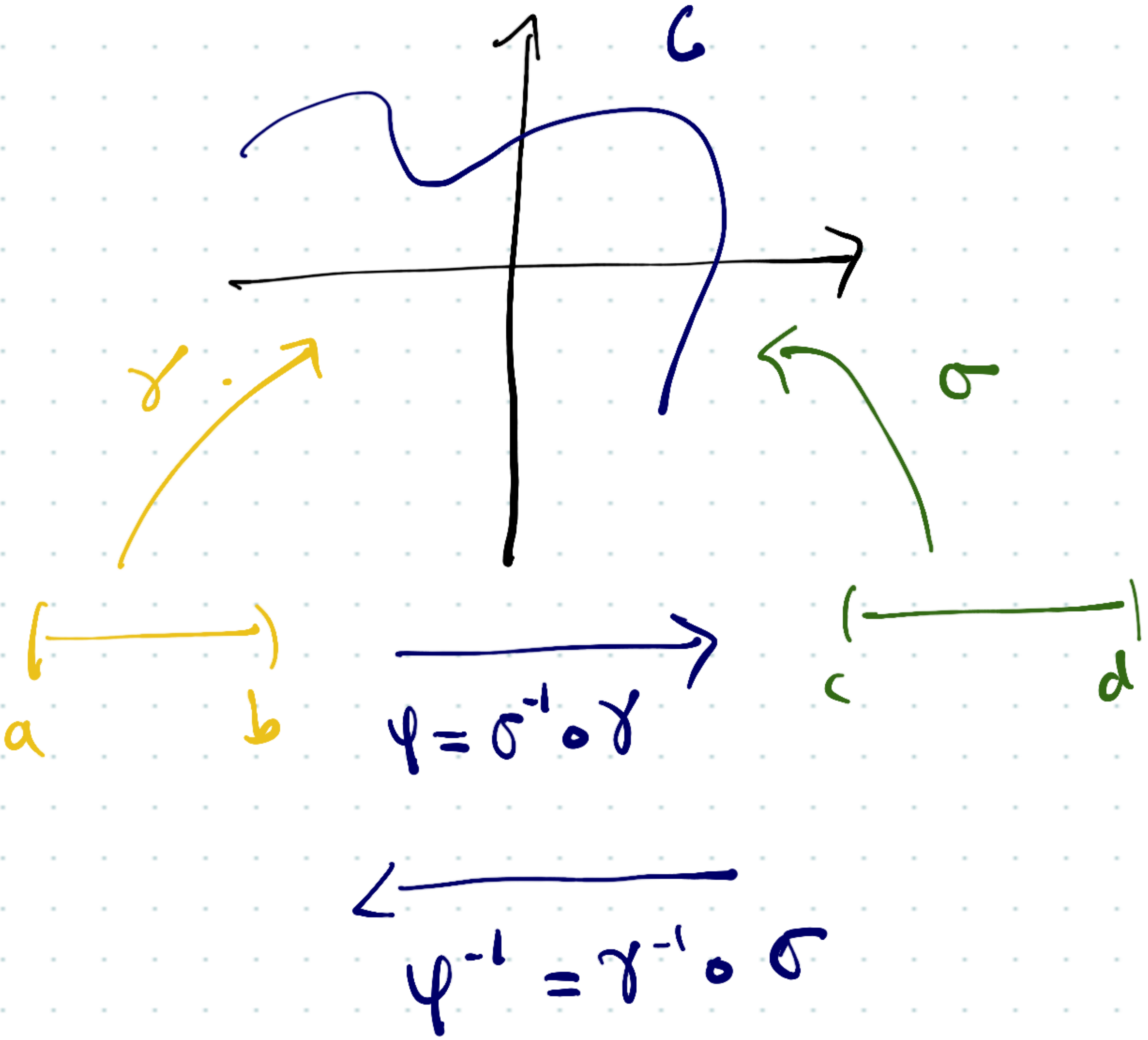


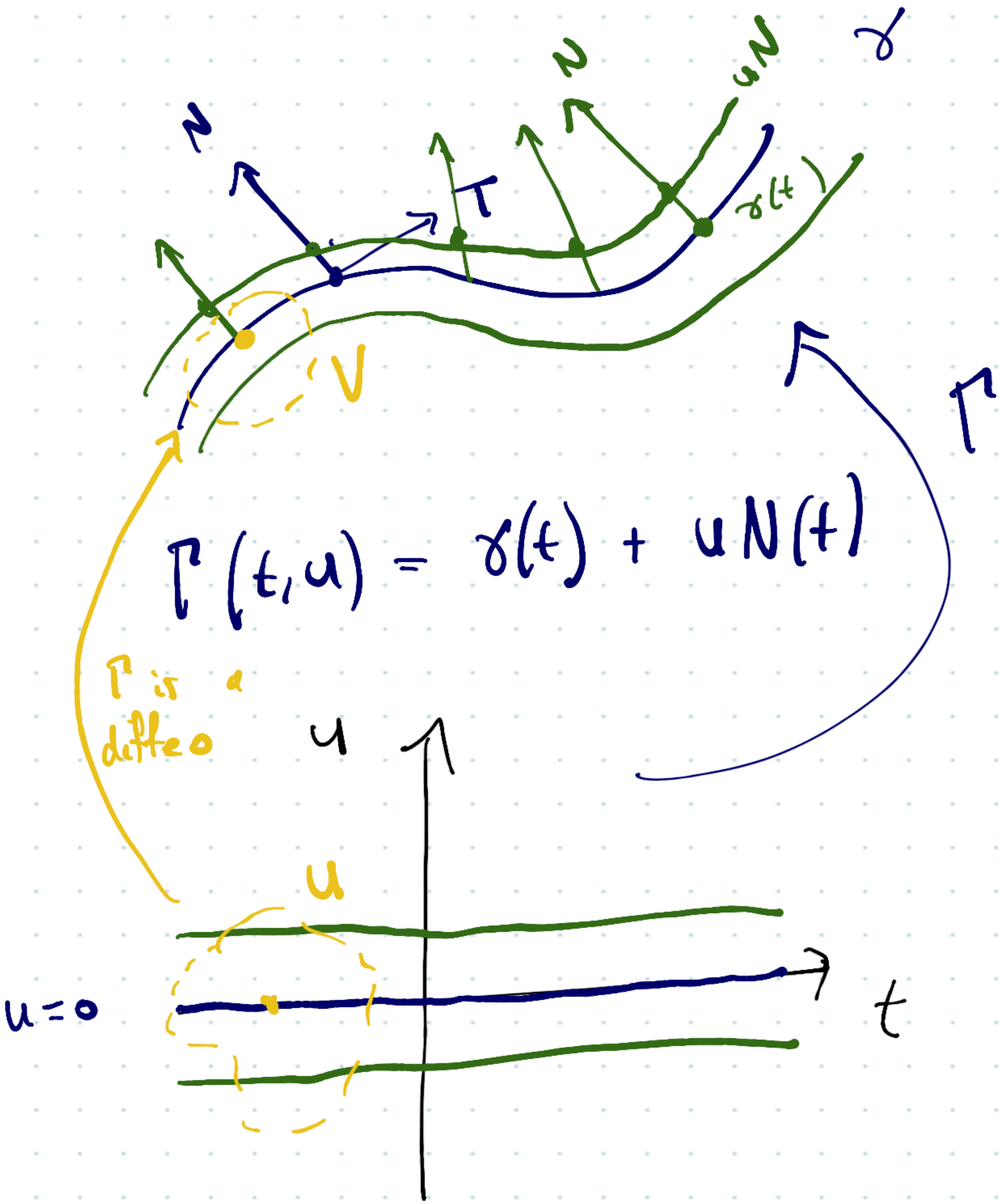
smoothly closes up



corner
doesn't close up
smoothly

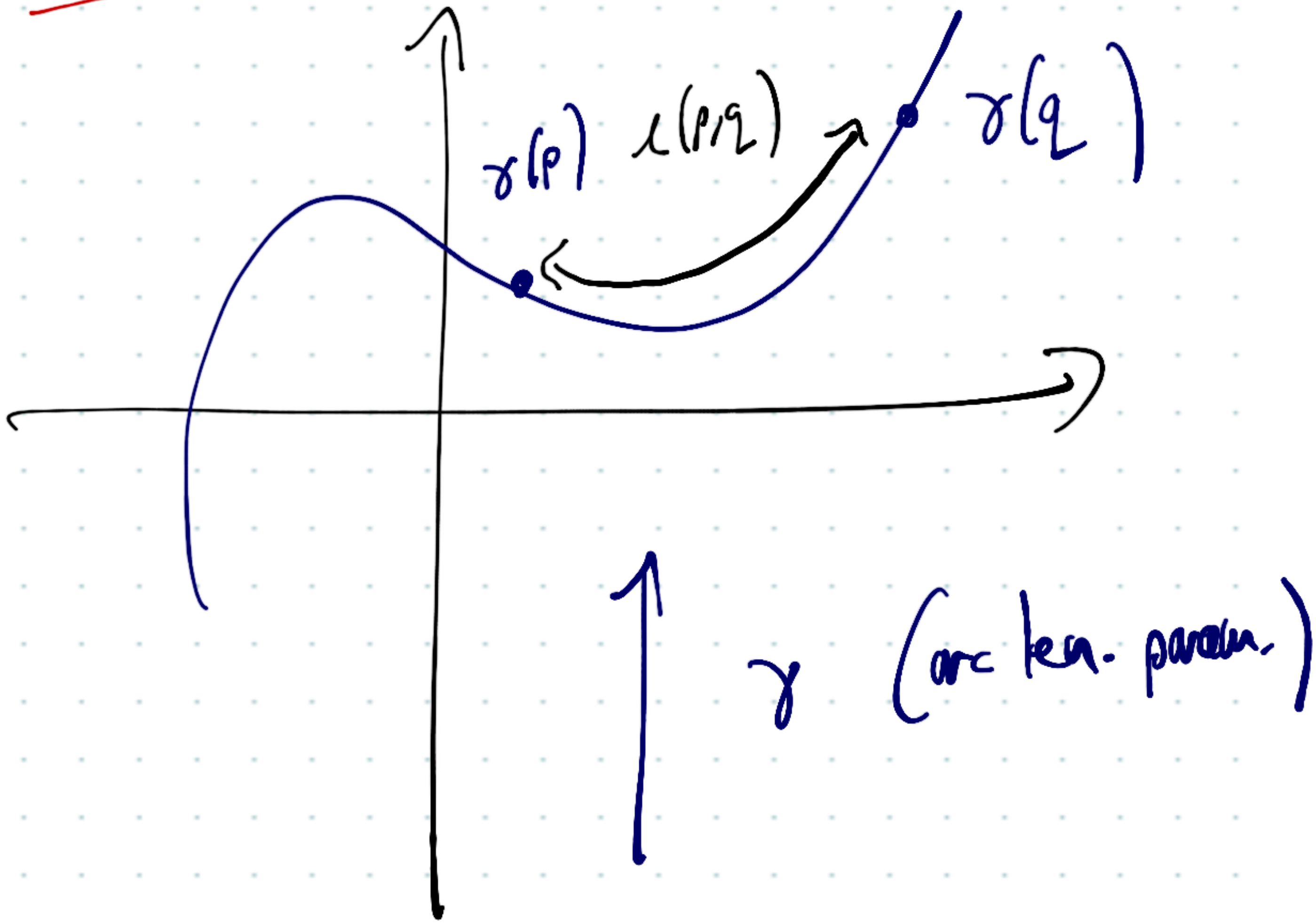
CHANGE OF PARAMETERS





N
N
uN
γ
T
$\gamma(t)$
V
Γ
$\Gamma(t,u) = \gamma(t) + uN(t)$
Γ is a diffeo
u
U
u=0
t

ARC LENGTH



In arclength : $|q-p| = l(p,q)$

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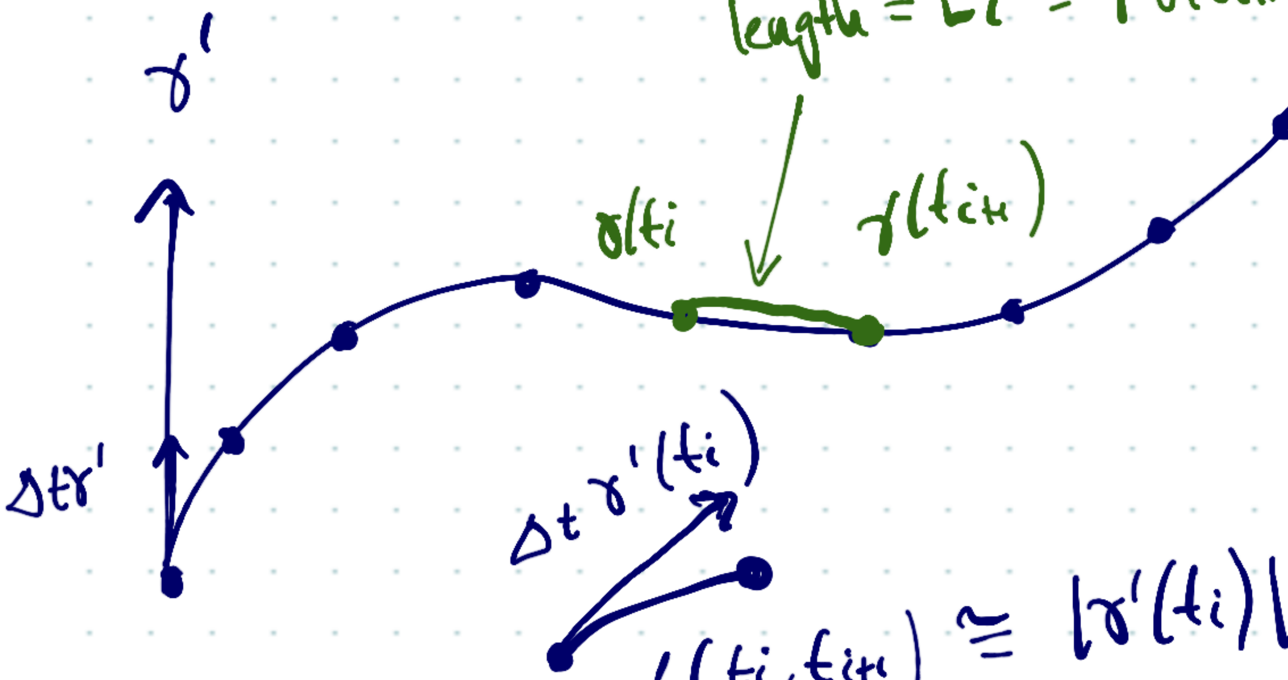
if $\varphi = \sigma^{-1} \circ \gamma$

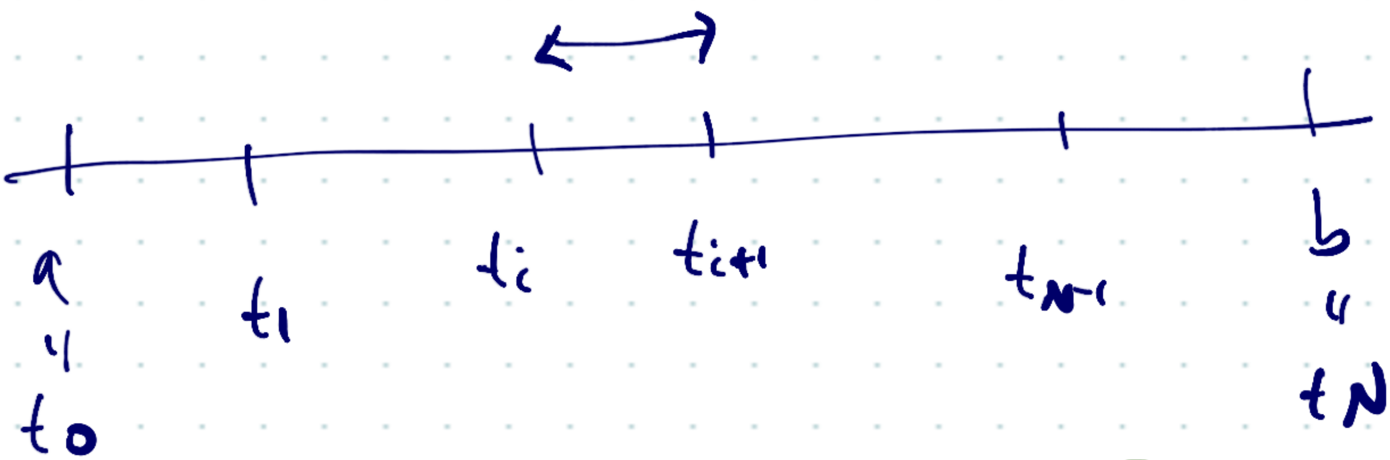
then $\sigma \circ \varphi = \sigma \circ \sigma^{-1} \circ \gamma = \gamma$

$$\text{length} = L_i = |\gamma(t_{i+1}) - \gamma(t_i)|$$

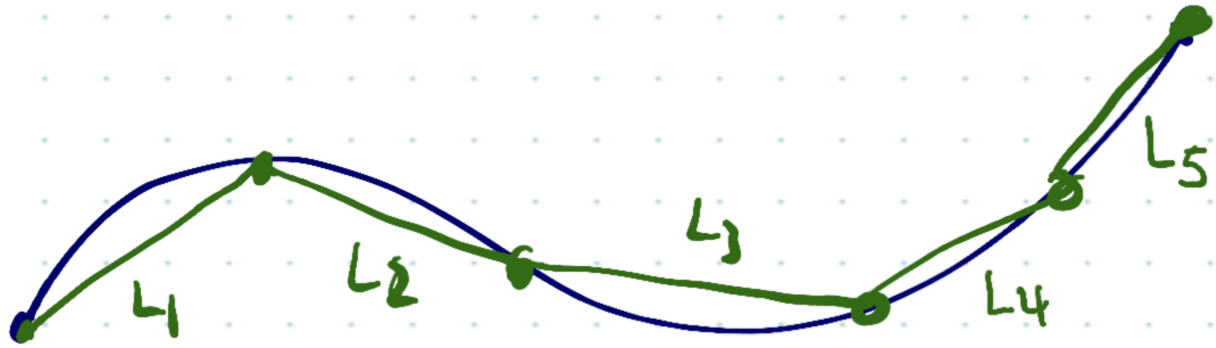


$$l(t_i, t_{i+1}) \cong |\gamma'(t_i)| \Delta t$$

$$\Delta t = \frac{b-a}{N}$$



$$\boxed{\text{Note: } \quad L_i = \left| \frac{\gamma(t_{i+1}) - \gamma(t_i)}{\Delta t} \right| \Delta t \xrightarrow[N \to \infty]{} |\gamma'| \, dt}$$

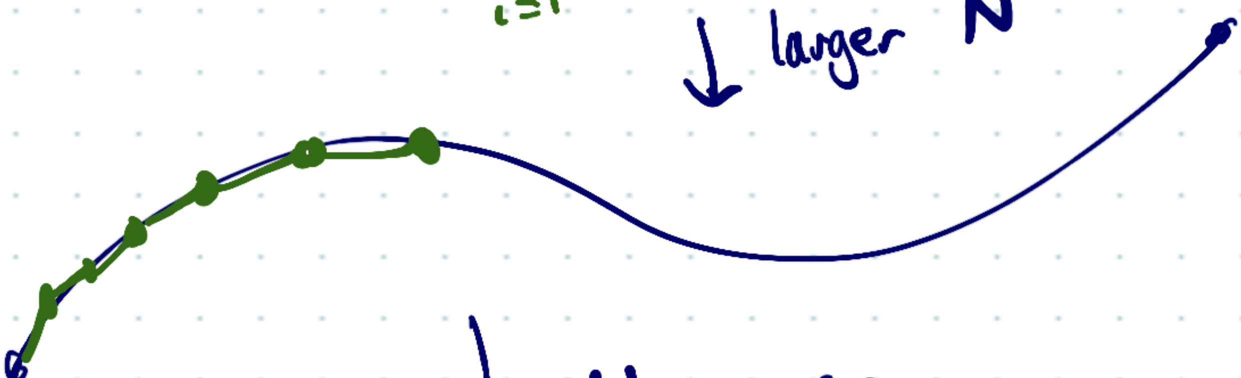


$$L(\gamma) = \lim_{N\to\infty} L(P_N)$$

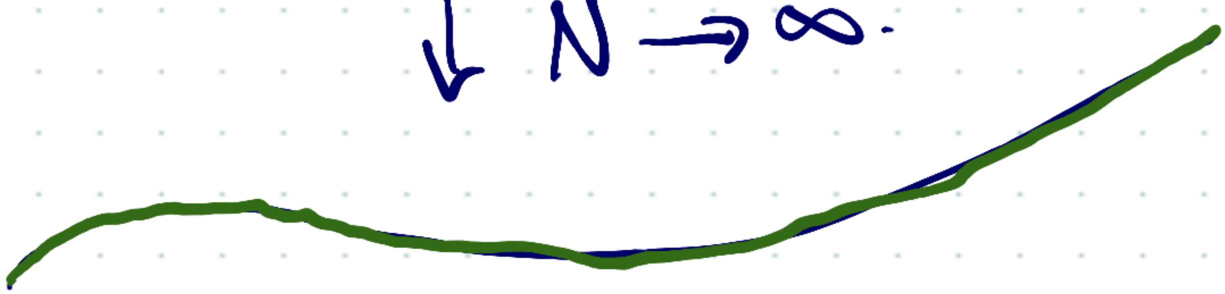
P_N = polygonal approximation

$$= \bigcup_{i=1}^{N} L_i$$

↓ larger N



↓ $N \to \infty$.



CURVATURE

Claim:

$$\partial_s \langle T, T\rangle = 2\langle \partial_s T, T\rangle$$

Proof: We prove if $X(s)$, $Y(s)$ are C^∞ (smooth)

Then

$$\partial_s \langle X, Y\rangle = \langle \partial_s X, Y\rangle + \langle X, \partial_s Y\rangle$$

PRODUCT RULE!

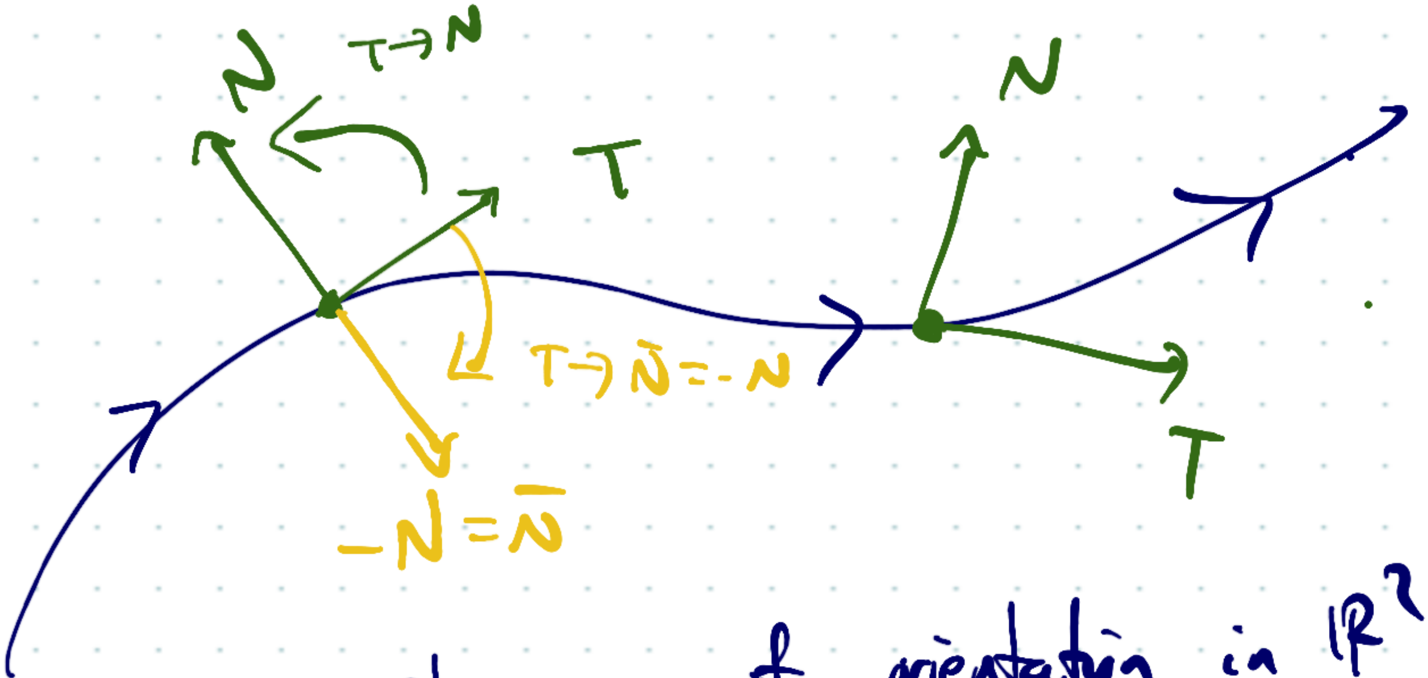
$$\partial_s (X\cdot Y) = (\partial_s X)\cdot Y + X\cdot(\partial_s Y)$$

write $X = (x_1, x_2)$, $Y = (y_1, y_2)$

$$X\cdot Y = x_1 y_1 + x_2 y_2 \quad \text{(defn)}$$

$$\partial_s (X\cdot Y) = \partial_s (x_1 y_1) + \partial_s (x_2 y_2)$$

$$\partial_s X\cdot Y + X\cdot \partial_s Y = (\partial_s x_1) y_1 + \partial_s (x_2)\cdot y_2 + x_1 (\partial_s y_1) + x_2 (\partial_s y_2)$$



Change of orientation in $\mathbb{R}^2$

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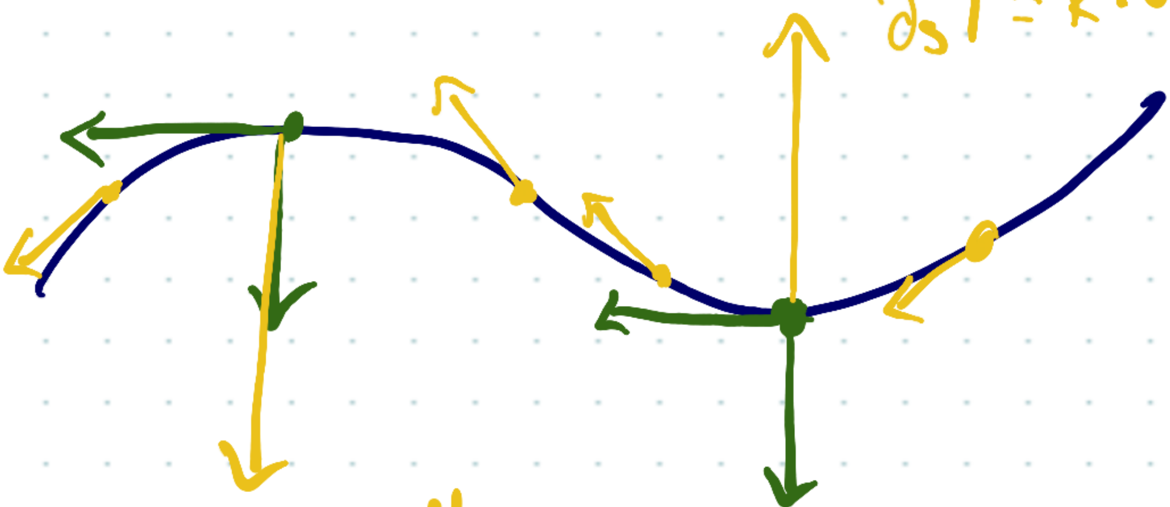
change of orientation of γ

sign of κ

"bends away from N"

$\partial_s T = \kappa N$, $\kappa < 0$



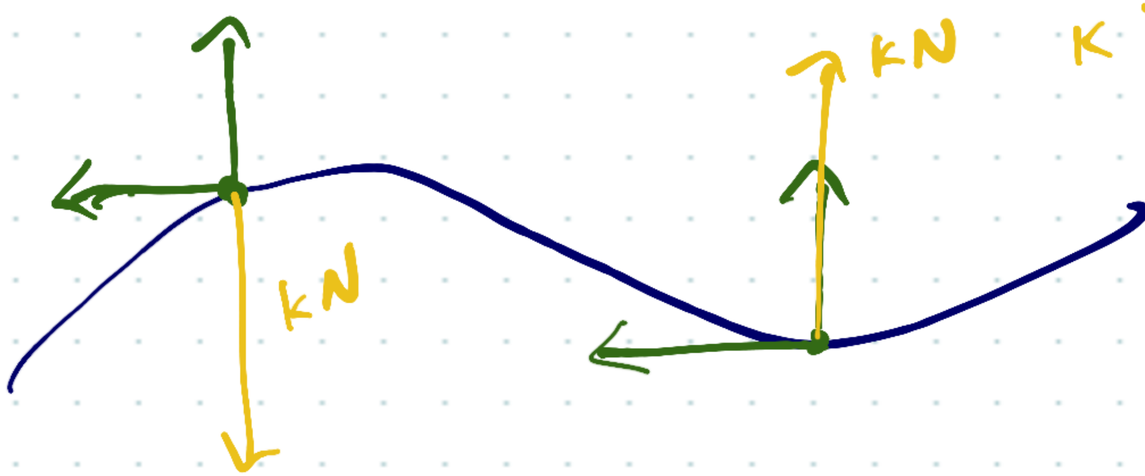
$\partial_s T = \kappa N$

"bends towards N"

$\kappa > 0$

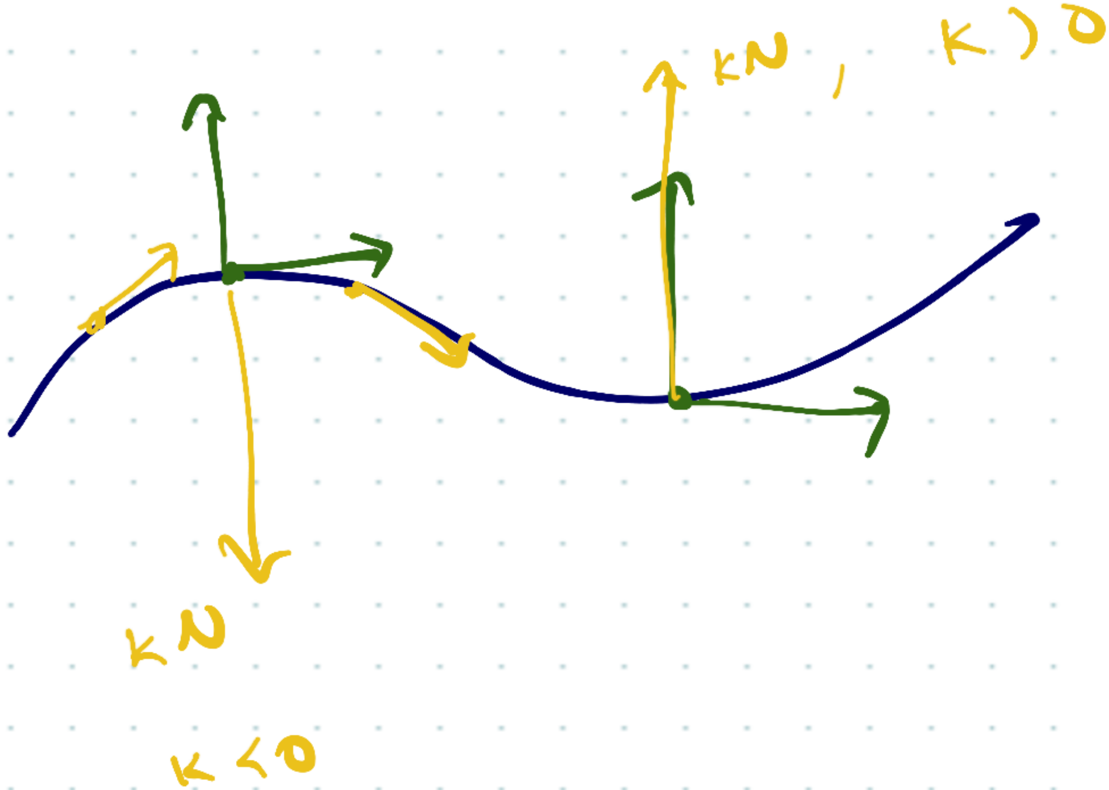
CHANGE $\mathbb{R}^2$ ORIENTATION CHANGES SIGN

κN $\kappa > 0$



$\kappa < 0$

CHANGE CURVE ORIENTATION
CHANGES SIGN



Note: $\vec{\kappa} := \kappa N$ is invariant under change of $\mathbb{R}^2$ or γ orientation