<u>claim:</u> $\forall \alpha, \beta$ $\tau_{\alpha\beta}$ is C^∞

$\Longrightarrow$ $\tau_{\alpha\beta}$ is a diffeomorphism

<u>Pf:</u> need to show

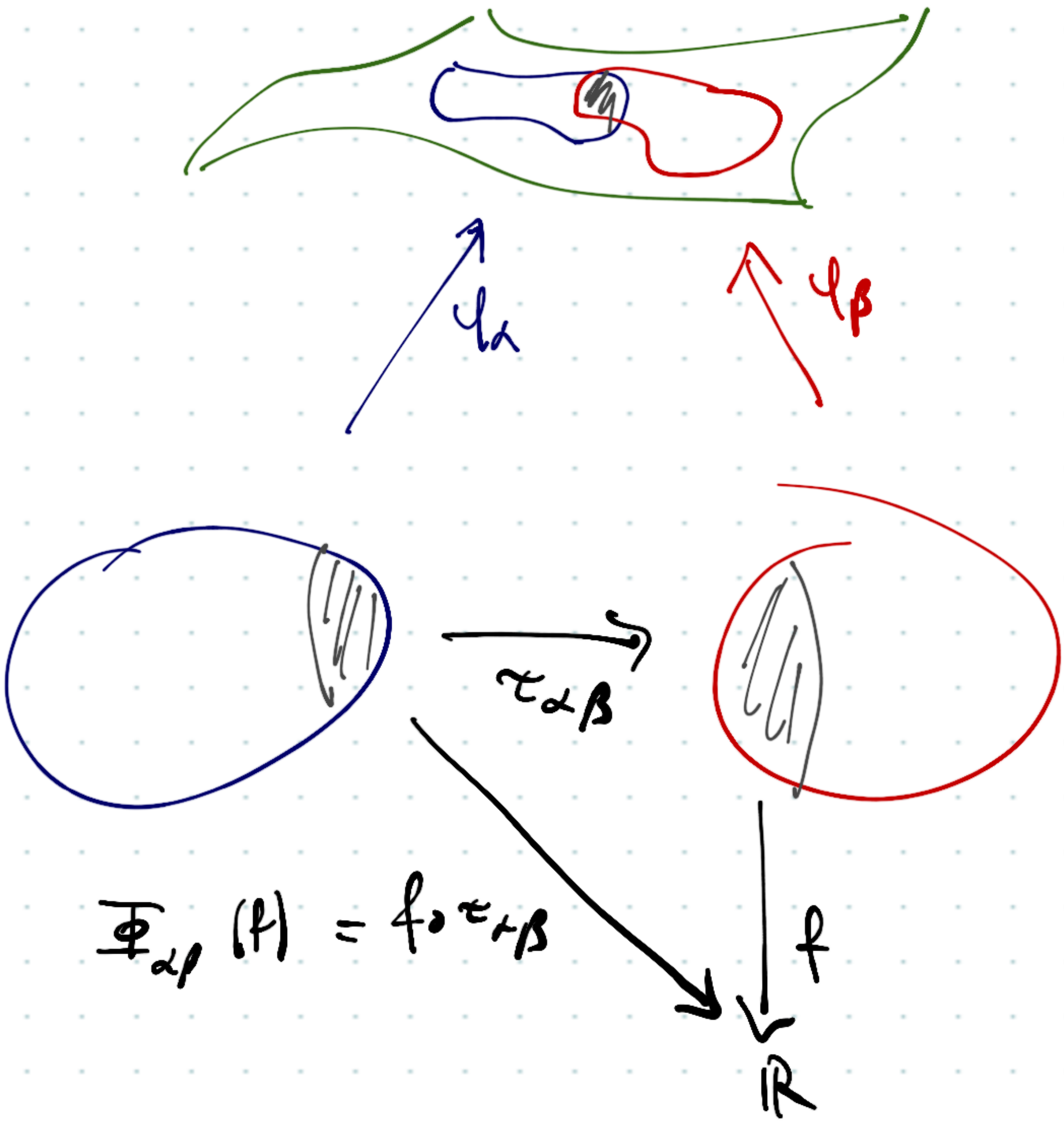
(i) $\tau_{\alpha\beta}^{-1}$ is defined

(ii) $\tau_{\alpha\beta}^{-1}$ is C^∞.

Now $\tau_{\alpha\beta} = \varphi_\beta^{-1} \circ \varphi_\alpha$

$$\tau_{\alpha\beta}^{-1} = (\varphi_\beta^{-1} \circ \varphi_\alpha)^{-1}$$
$$= \varphi_\alpha^{-1} \circ \varphi_\beta \quad \text{(i)}$$
$$= \tau_{\beta\alpha} \text{ is } C^\infty \text{ by assumption}$$

(ii) $\boxtimes$



$$\Phi_{\alpha\beta}(f) = f \circ \tau_{\alpha\beta}$$

$$\Phi_{\alpha\beta}^{-1} = \Phi_{\beta\alpha}$$

$$\begin{aligned}
\Phi_{\alpha\beta} \circ \Phi_{\beta\alpha}(f) &= \Phi_{\alpha\beta}\left(f \circ \tau_{\beta\alpha}\right) \\
&= \left(f \circ \tau_{\beta\alpha}\right) \circ \tau_{\alpha\beta} \\
&= f \circ \left(\tau_{\beta\alpha} \circ \tau_{\alpha\beta}\right) = f
\end{aligned}$$

Eg 1 suppose

$$\tau(u,v) = (u^2, v^2) = (x,y)$$

The function $f(x,y) = x+y$

has $df = \begin{pmatrix} 1 & 1 \end{pmatrix}$

But $f \circ \tau(u,v) = u^2 + v^2$

has $d(f\circ\tau) = \begin{pmatrix} 2u & 2v \end{pmatrix}$

Then $d(f\circ\tau)\big|_{(0,0)} = 0$

i.e. $(u,v) = (0,0)$ is a critical point.

But $df\big|_{\tau(0,0)} = df\big|_{(0,0)}$

$$= \begin{pmatrix} 1 & 1 \end{pmatrix}$$

$$\neq 0$$

Note $d\tau = \begin{pmatrix} 2u & 0 \\ 0 & 2v \end{pmatrix} \rightsquigarrow d\tau_{(0,0)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

<u>Check</u> : $\varphi^{-1} = \pi|_{\mathrm{Gr} f}$ is cts.

<u>Pf</u>: let $(x_n) \subseteq \mathrm{Gr} f$
s.t. $x_n \to x \in \mathrm{Gr} f$

need to show:

$$\lim_{n\to\infty} \varphi^{-1}(x_n) = \varphi^{-1}\left(\lim_{n\to\infty} x_n\right)$$
$$= \varphi^{-1}(x)$$

But π is cts.

$$\therefore \quad \begin{array}{ccc} \displaystyle\lim_{n\to\infty} \pi(x_n) & = & \pi(x) \\ \| & & \| \\ \displaystyle\lim_{n\to\infty} \varphi^{-1}(x_n) & & \varphi^{-1}(x) \end{array}$$

$\boxtimes$

Claim: $df = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \partial_u f & \partial_v f \end{pmatrix}$ is injective.

Pf: suffices to prove

$$(*) \quad df \cdot \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow x^1 = 0, \; x^2 = 0.$$

since then if $df\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = df\begin{pmatrix} y^1 \\ y^2 \end{pmatrix}$

$$\Rightarrow df \cdot \left[\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} - \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} \right] = 0$$

$$(*) \Rightarrow \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} - \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

ie $\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} y^1 \\ y^2 \end{pmatrix}$

Then

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = df \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \partial_u f & \partial_v f \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} x^1 \\ x^2 \\ x^1 \partial_u f + x^2 \partial_v f \end{pmatrix}$$

$$\Rightarrow x^1 = 0, \; x^2 = 0 \quad \boxtimes$$

claim: $df = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \partial_u f & \partial_v f \end{pmatrix}$ is injective

pf: rank $df = 2$

By rank nullity thm ...

use previous page

$\Rightarrow$ suffices to show

$\dim \ker = 0$

Recall for graphs we defined

$$G(u,v,w) = (u, v, f(u,v) + w)$$

showed dG is non-singular (since $d\varphi$ is injective)

$$\Rightarrow G^{-1} \text{ is } C^\infty.$$

Note that $G^{-1}|_{\Gamma f} = \varphi^{-1}$

where $\varphi(u,v) = (u, v, f(u,v))$

$$p = (x_0, y_0, z_0)$$

$$N = (N^1, N^2, N^3)$$

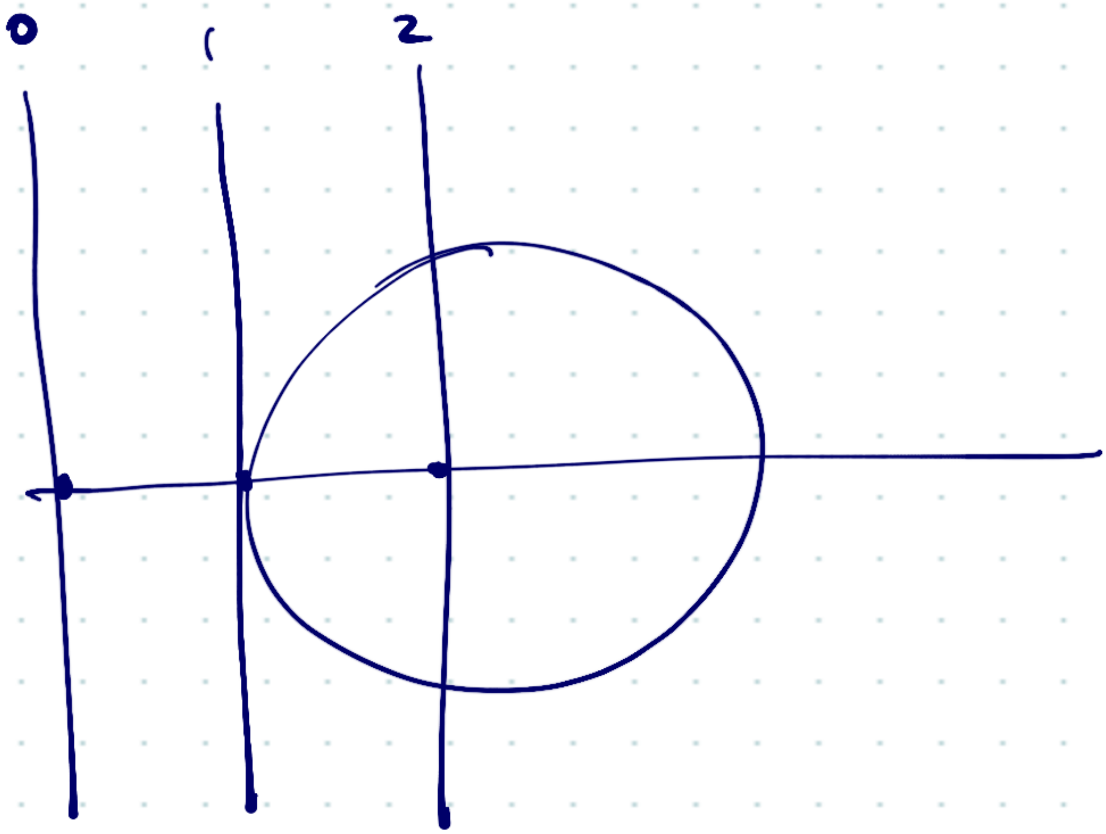
$$p + tN = (x_0 + tN^1, y_0 + tN^2, z_0 + tN^3)$$

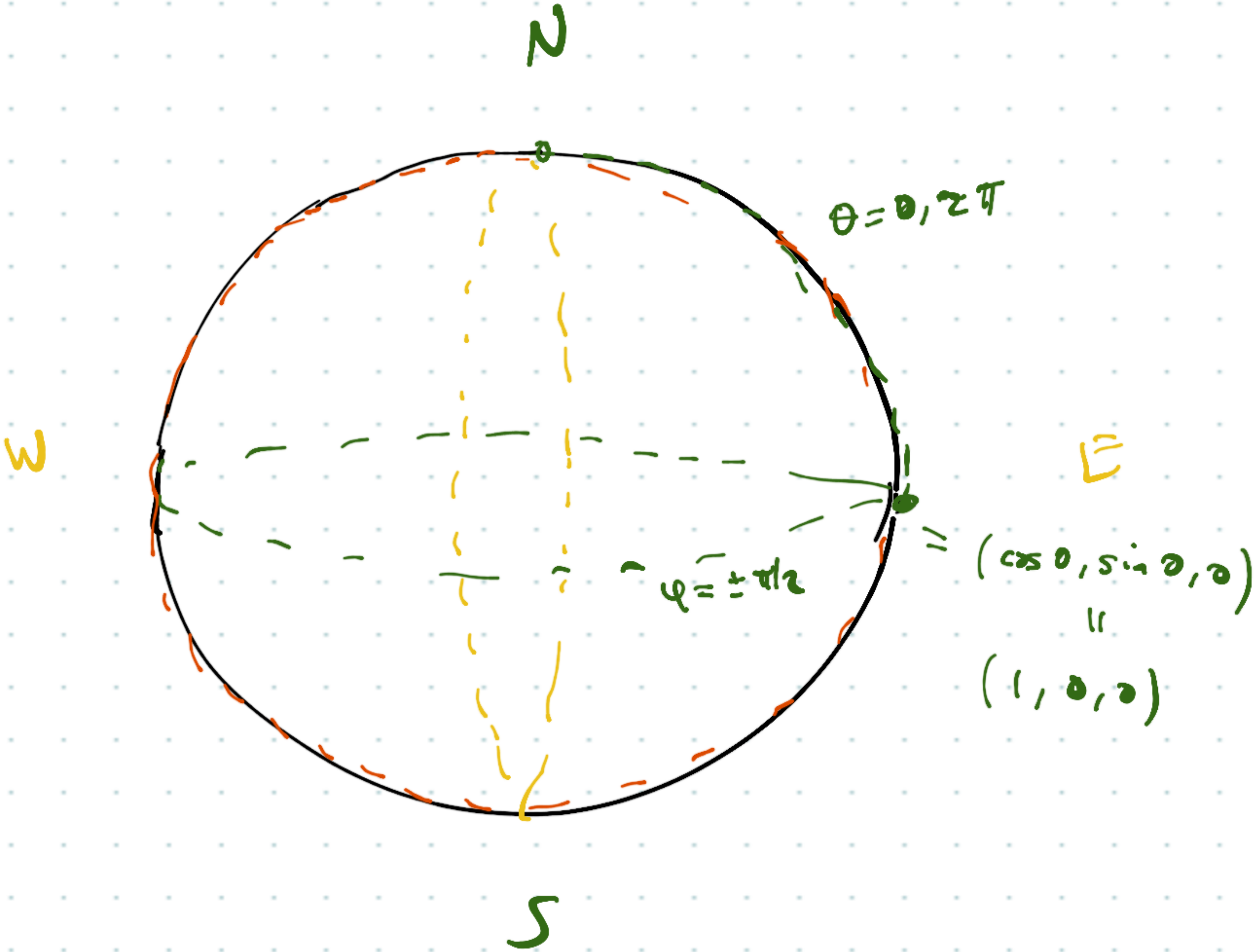
$$\in S^2 = \{x^2 + y^2 + z^2 = 1\}$$

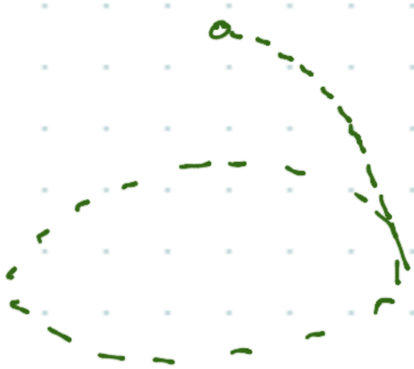
iff $(x_0 + tN^1)^2 + (y_0 + tN^2)^2 + (z_0 + tN^3)^2 - 1 = 0$

$$\parallel$$

$$at^2 + b^? + c = 0$$



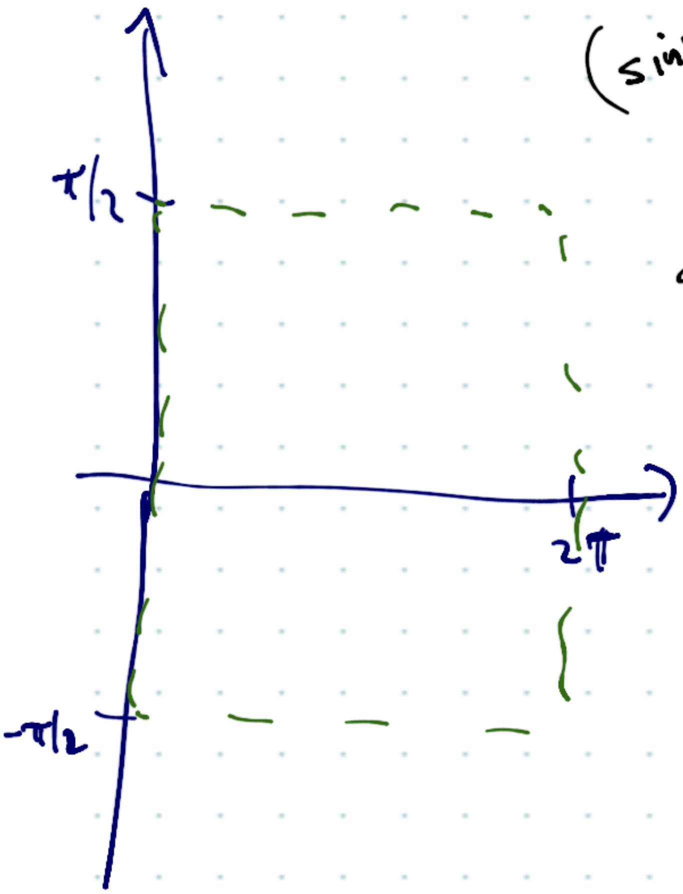




missing part of N

2/ φ_N is a homeo

$(\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$



$$S_N = S^n \cap \left[\{ z > 0 \} \cup \{ y \neq 0, x > 0 \} \right]$$

open.

$$(\theta, \varphi) = \varphi^{-1}(x, y, z) = \left(\operatorname{atan2}\left(\tfrac{y}{x}\right), \arccos z \right)$$

see wikipedia for atan2 defn.

(x) note φ_N is injective

$$(\sin\varphi_1 \cos\theta_1, \sin\varphi_1 \sin\theta_1, \cos\varphi_1) = (\sin\varphi_2 \cos\theta_2, \sin\varphi_2 \cos\theta_2, \cos\varphi_2)$$

$\Rightarrow \cos\varphi_1 = \cos\varphi_2 \Rightarrow \varphi_1 = \varphi_2$. Then $\left.\begin{matrix} \cos\theta_1 = \cos\theta_2 \\ \sin\theta_2 = \sin\theta_2 \end{matrix}\right\} \Rightarrow \theta_1 = \theta_2$

3/ $d\varphi_N$ is injective

$$d\varphi_N = \begin{pmatrix} -\sin\varphi \sin\theta & \cos\varphi\cos\theta \\ \sin\varphi\cos\theta & \cos\varphi \sin\theta \\ 0 & -\sin\varphi \end{pmatrix}$$

$$\det \begin{pmatrix} -\sin\varphi\sin\theta & \cos\varphi\cos\theta \\ \sin\varphi\cos\theta & \cos\varphi\sin\theta \end{pmatrix}$$

$$= -\sin^2\theta\cos\varphi\sin\varphi - \cos^2\theta\sin\varphi\cos\varphi$$

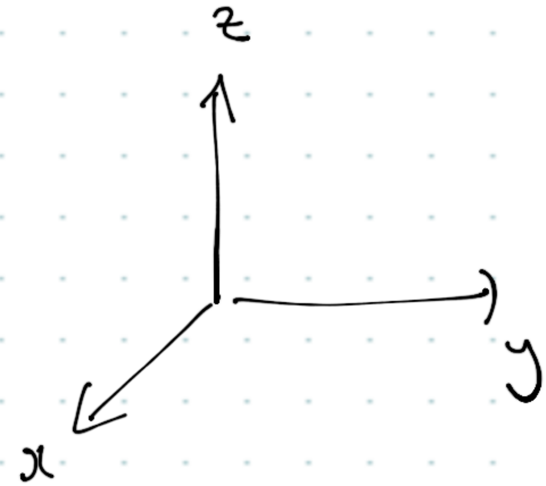
$$= -2\sin\varphi\cos\varphi \neq 0 \quad \text{for } \varphi \in (0, \pi/2)$$

2/ Observe φ_N is injective & $d\varphi_N$ is injective

It is not always true then that φ_N is a homeo

e.g.

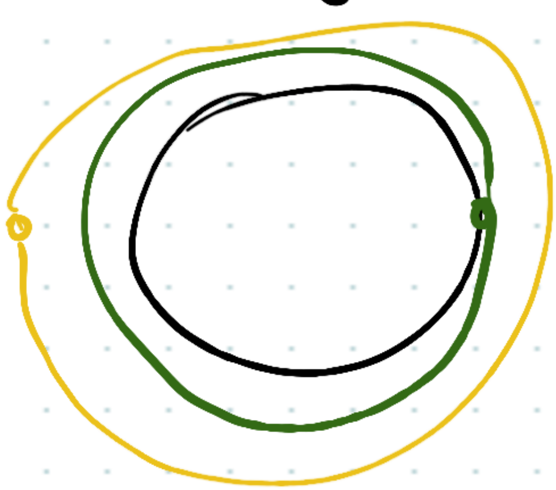
φ is injective w/ $d\varphi$ injective but φ^{-1} is not cts!



$T^2 \cong S^1 \times S^1$

$\theta \, , \, \varphi$

cover S^1 by u v



cover T^2 by

$$\left.\begin{array}{l} u \times u \\ u \times v \\ v \times u \\ v \times v \end{array}\right\} 4$$