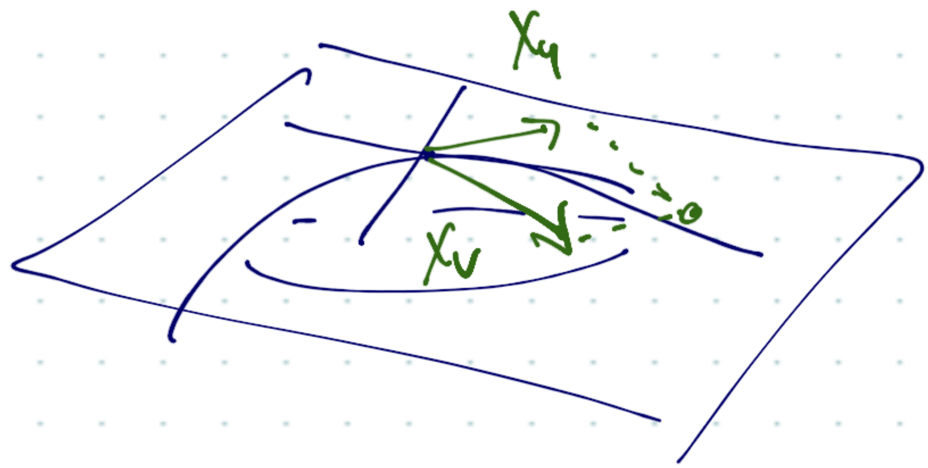
if $y = f(x)$, $x = f^{-1}(y)$

what is $\left[f^{-1}(y)\right]'$

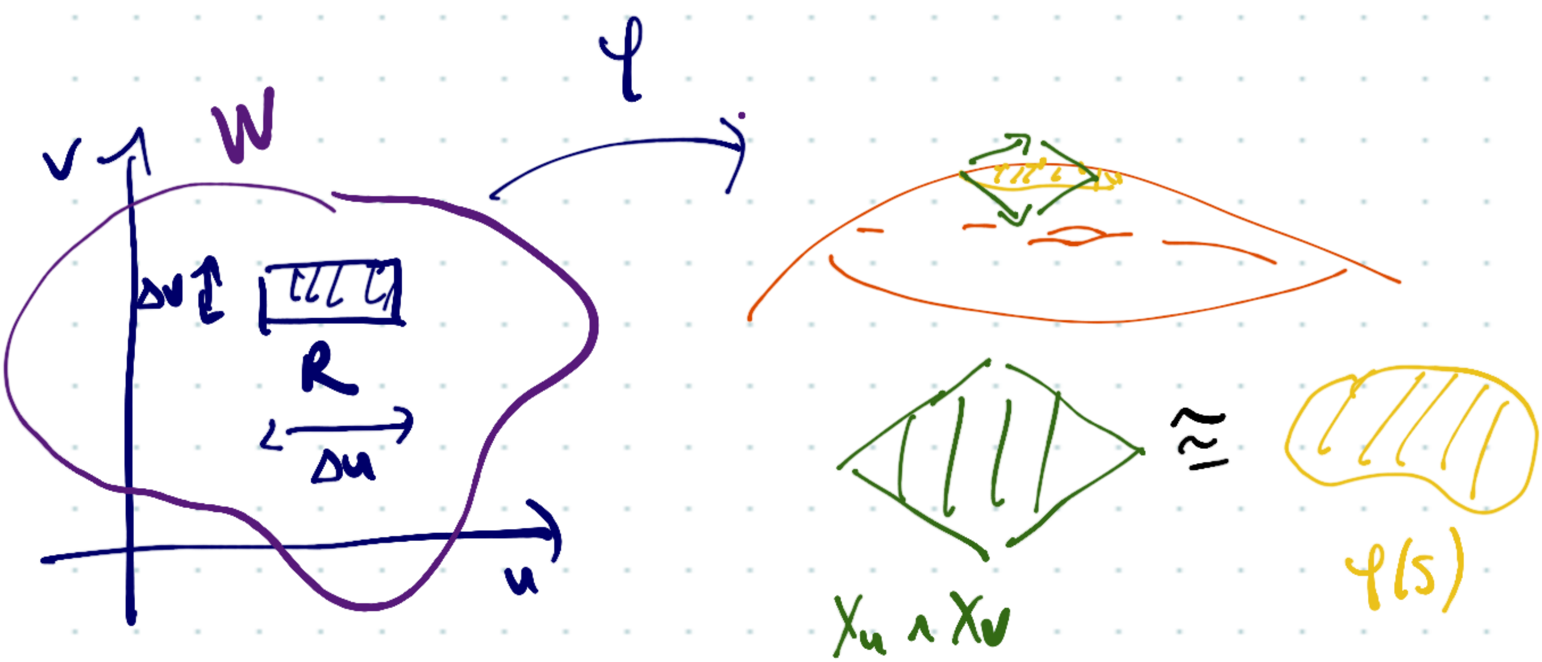
$$= \frac{1}{f'(f^{-1}(y))}$$

follow from:

$f \circ f^{-1} = \mathrm{Id}$ + chain rule.


Xu
Xv


Xu ∧ Xv


φ
v
W
Δv
R
Δu
u
Xu ∧ Xv
≈
φ(S)

$$g = \begin{pmatrix} \langle X_u, X_u \rangle & \langle X_u, X_v \rangle \\ \langle X_v, X_u \rangle & \langle X_v, X_v \rangle \end{pmatrix}$$

$$= \begin{pmatrix} g_{uu} & g_{uv} \\ g_{vu} & g_{vv} \end{pmatrix}$$

functions of (u,v)

$$X_u = \partial_u \varphi, \qquad X_v = \partial_v \varphi$$

Example: $\mathbb{S}^2$

$$f(\theta, \varphi) = (\cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi)$$

$$X_\theta = -\sin\theta \sin\varphi\, e_x + \cos\theta \sin\varphi\, e_y$$

$$X_\varphi = \cos\theta\cos\varphi\, e_x + \sin\theta \cos\varphi\, e_y - \sin\varphi\, e_z$$

$$e_x = (1,0,0), \quad e_y = (0,1,0), \quad e_z = (0,0,1)$$

$$g_{\theta\theta} = \langle X_\theta, X_\theta \rangle_{\mathbb{R}^3}$$

$$= |X_\theta|^2_{\mathbb{R}^3}$$

$$= \sin^2\theta \sin^2\varphi + \cos^2\theta \sin^2\varphi = \sin^2\varphi$$

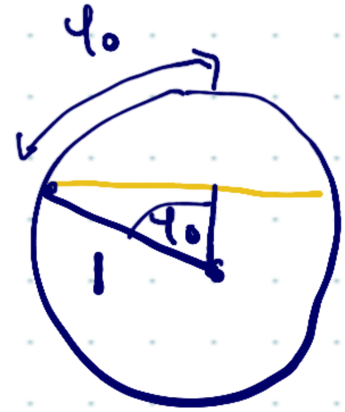
$$g_{\varphi\varphi} = \cos^2\theta \cos^2\varphi + \sin^2\theta \cos^2\varphi + \sin^2\varphi$$

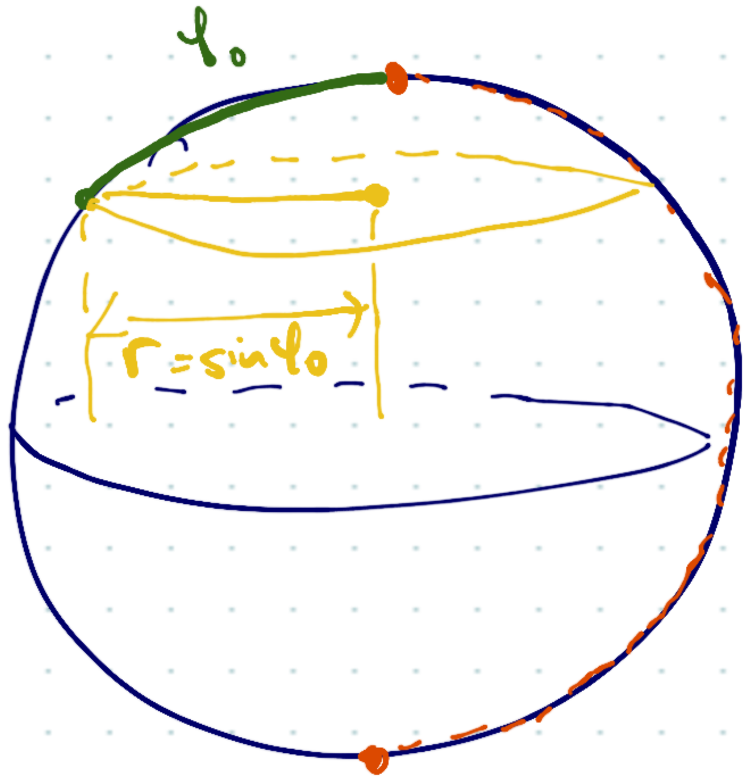
$$= \cos^2\varphi + \sin^2\varphi$$

$$= 1$$

check: $g_{\theta\varphi} = g_{\varphi\theta} = \langle X_\theta, X_\varphi \rangle = 0$

$$g_{S^2} = \begin{pmatrix} \sin^2\varphi & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{array}{l} 0 < \theta < 2\pi \\ 0 < \varphi < \pi \end{array}$$





$$L(\gamma) = \int_0^{2\pi} |X_\theta(t, \varphi_0)|\, dt$$

$$= 2\pi \sin\varphi_0$$

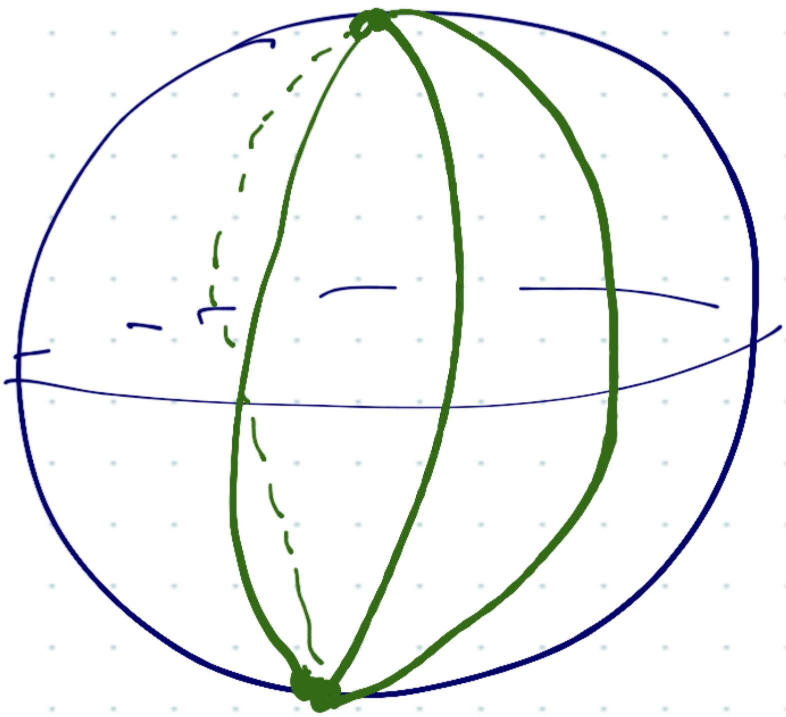
$$\gamma(t) = f(t, \varphi_0)$$

$$\gamma' = \partial_\theta f = X_\theta$$

$$|X_\theta|^2 = g(X_\theta, X_\theta) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \sin^2\varphi_0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \sin^2\varphi_0 \\ 0 \end{pmatrix} = \sin^2\varphi_0$$

φ – coord curves

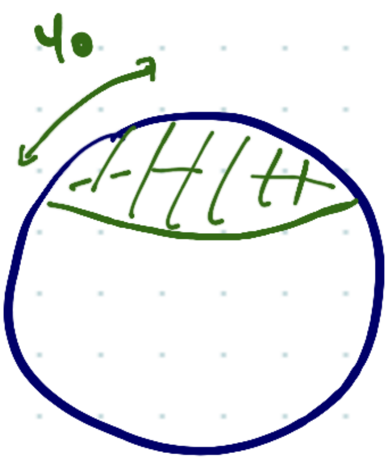


arcs of great circles

Remark:

$$\det g = \begin{vmatrix} \sin^2\varphi & 0 \\ 0 & 1 \end{vmatrix}$$

$$= \sin^2\varphi$$



$$\text{Area} = \int_0^{2\pi}\int_0^{\varphi_0} \sqrt{\det g}\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\varphi_0} |\sin\varphi|\, d\varphi\, d\theta$$

$$= 2\pi(1 - \cos\varphi_0)$$

$\mathcal{E} = (e_1, e_2)$ $\qquad$ $A_{\mathcal{E}\mathcal{F}}(e_1) = e_1$

$\mathcal{F} = (e_1, e_1 + e_2)$ $\qquad$ $A_{\mathcal{E}\mathcal{F}}(e_2) = e_1 + e_2$

$$A_{\mathcal{E}\mathcal{F}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

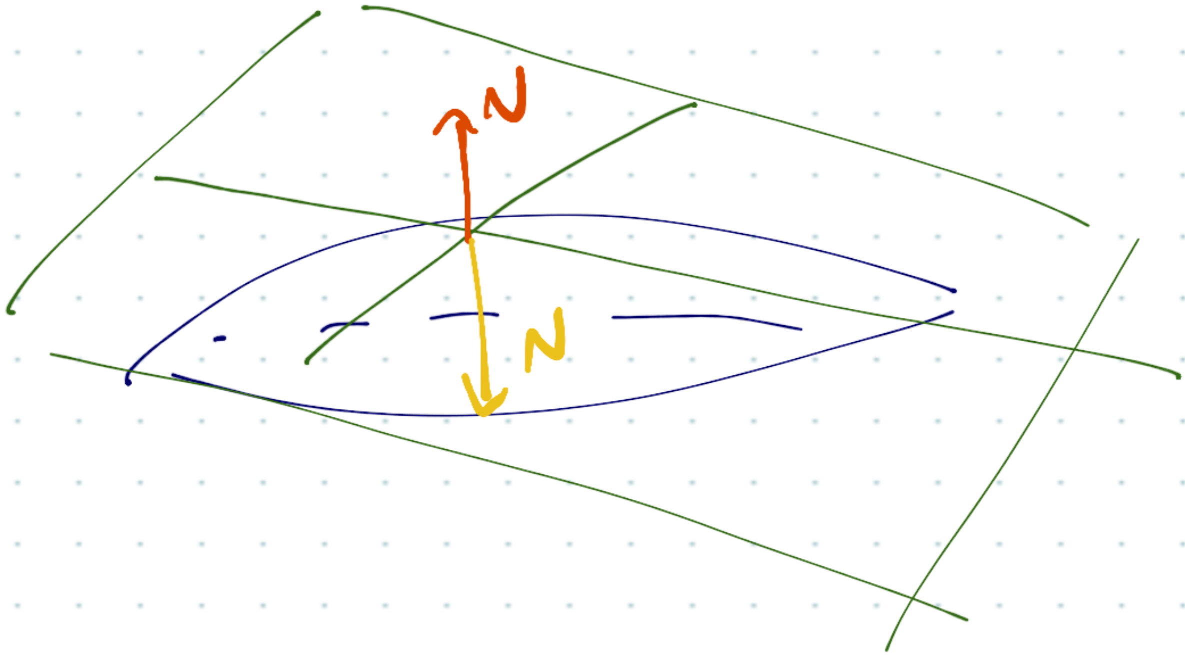
$$\det A_{\mathcal{E}\mathcal{F}} = 1 > 0$$

$$\mathcal{E} \sim \mathcal{F}$$

$\mathcal{E} = (e_1, e_2)$, $\mathcal{F} = (e_2, e_1)$

$A_{\mathcal{E}\mathcal{F}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\det A_{\mathcal{E}\mathcal{F}} = -1 < 0$

$$\mathcal{E} \not\sim \mathcal{F}$$


N
N

Normal to graph

$$X_x = (1, 0, \partial_x f)$$

$$X_y = (0, 1, \partial_y f)$$

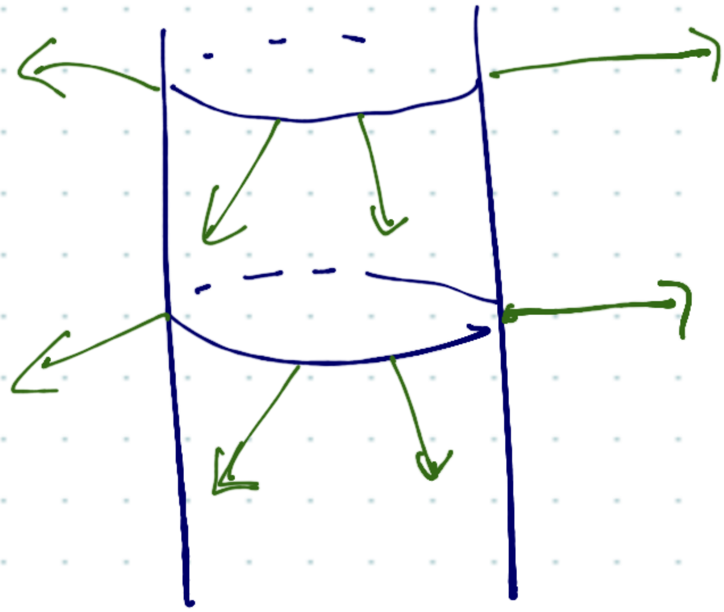
then $(-\partial_x f, -\partial_y f, 1) \perp X_x, X_y$.

In general: solve

$$\left.\begin{aligned} N \cdot X_u &= 0 \\ N \cdot X_v &= 0 \end{aligned}\right\}$$ system of 2 linear equations in 3 unknowns N_1, N_2, N_3.

$$\begin{pmatrix} X_u \\ X_v \end{pmatrix} \cdot N = 0$$

rank $= 2$ $\implies$ $\dim \ker = 3 - \text{rank}$
$= 3 - 2 = 1.$



$$dN_p \cdot X = \frac{d}{dt}\Big|_{t=0} N(p + tX)$$

Define $\quad F(V) = A \cdot V$

$$dF_V \cdot W = \frac{d}{dt}\Big|_{t=0} F(V + tW)$$

$$= \frac{d}{dt}\Big|_{t=0} \left[A \cdot V + t\, A \cdot W \right]$$

$$= A \cdot W$$

$$\langle T, N \rangle = 0$$

$$\Rightarrow 0 = \partial_s \langle T, N \rangle$$

$$= \langle \partial_s T, N \rangle + \langle T, \partial_s N \rangle$$