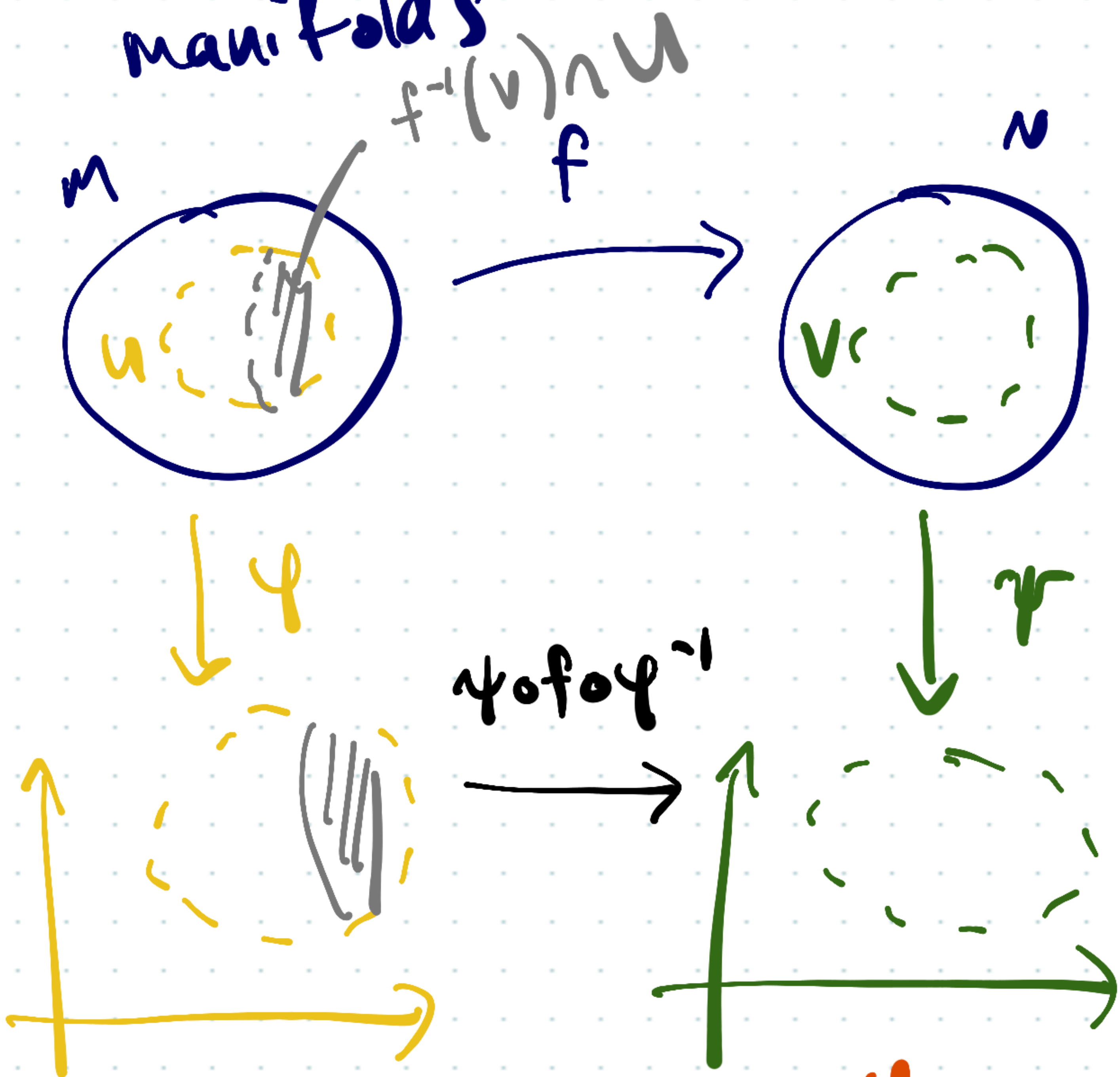
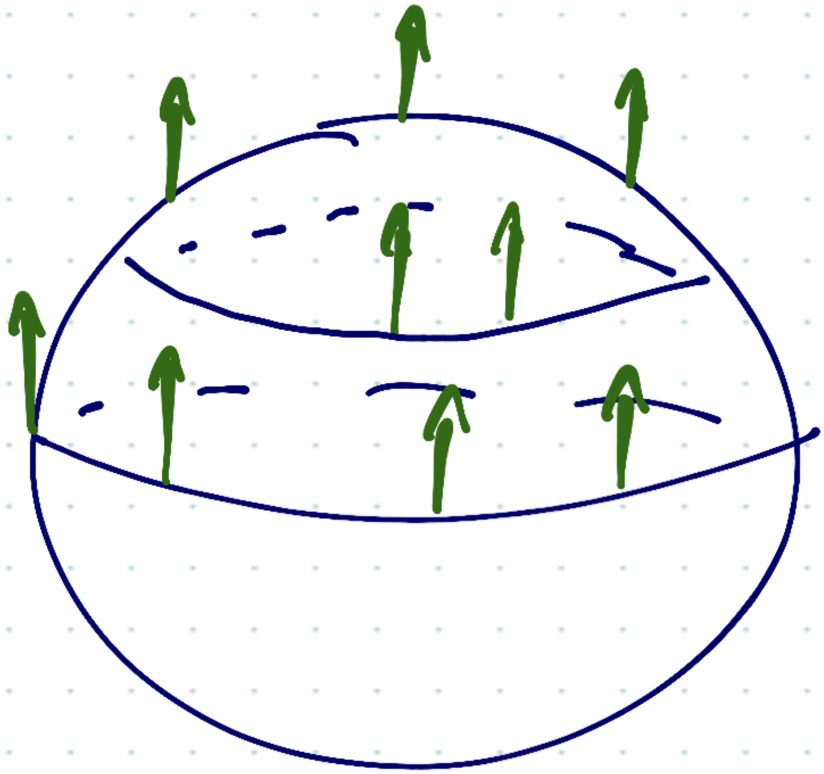
Smooth maps between manifolds



<u>DEFN</u> f is smooth if $\forall$ charts $\psi \circ f \circ \varphi^{-1}$ is smooth



$e_1 = (1,0,0)$

$X = e_1 - \langle e_1, N\rangle N$

check X is tangent:

$\langle X, N\rangle = \langle e_1 - \langle e_1, N\rangle N, N\rangle$
$= \langle e_1, N\rangle - \langle e_1, N\rangle\langle N, N\rangle = 0.$

$df_x \cdot V$ is independent of

γ s.t. $V = [\gamma]$

i.e. $\gamma(0) = \pi(V) = x$

$\gamma'(0) = V$

If $[\gamma] = [\sigma] = V$

i.e. $\gamma(0) = \sigma(0)$

$\gamma'(0) = \sigma'(0)$

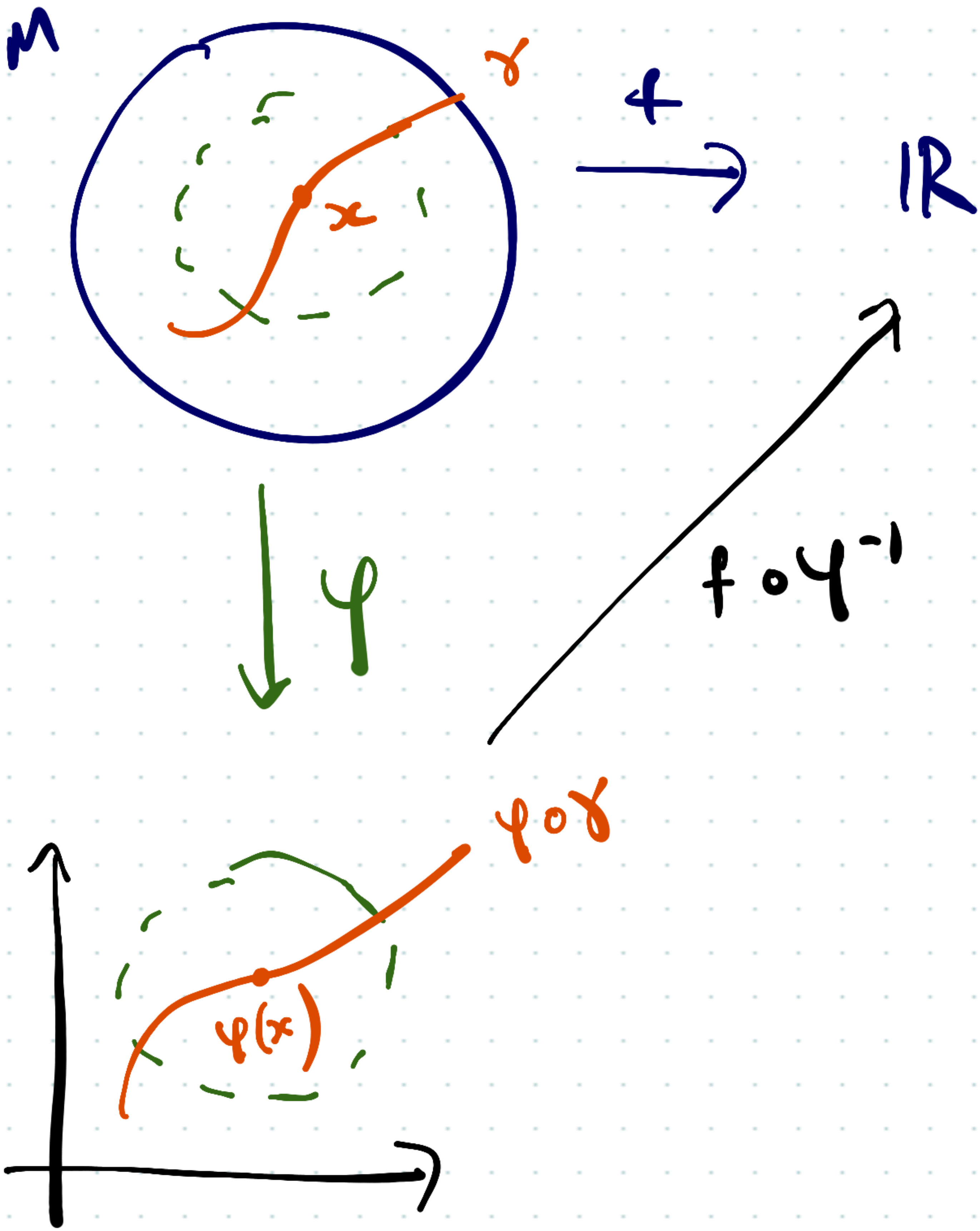
By the chain rule

$$\partial_t \big|_{t=0} f(\gamma(t)) = df_{\gamma(0)} \cdot \gamma'(0)$$

$$= df_{\sigma(0)} \cdot \sigma'(0)$$

$$= \partial_t \big|_{t=0} f(\sigma(t))$$

$\boxtimes$



M
γ
x
f
ℝ
φ
f∘φ⁻¹
φ∘γ
φ(x)

note

$$f(\gamma(t)) = f \circ (\varphi^{-1} \circ \varphi) \circ \gamma\ (t)$$

$$= \underbrace{(f \circ \varphi^{-1})}_{\text{is } C^\infty} \circ \underbrace{(\varphi \circ \gamma)}_{\text{is } C^\infty}\ (t)$$

$$\Longrightarrow f(\gamma(t)) \text{ is } C^\infty.$$

$$X = y\partial_x$$

$$Y = \partial_y$$

$$[X, Y]f = \partial_X \partial_Y f - \partial_Y \partial_X f$$

$$= \partial_{y\partial_x} \partial_{\partial_y} f - \partial_{\partial_y} \partial_{y\partial_x} f$$

$$= y\partial_x(\partial_y f) - \partial_y(y\partial_x f)$$

$$= y\partial_x\partial_y f - (\partial_x f + y\partial_y\partial_x f)$$

$$= y(\partial_x\partial_y f - \partial_y\partial_x f) - \partial_x f$$

$$= -\partial_x f$$

$$\nabla_{\partial_u}(Y^u \partial_u) = dY^u(\partial_u)\partial_u + Y^u \nabla_{\partial_u}\partial_u$$

$$= (\partial_u Y^u)\partial_u + Y^u \nabla_{\partial_u}\partial_u$$