MATH704 Differential Geometry

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Lecture Five: Surfaces that are graphs

Lecture Five: Surfaces that are graphs - Smooth functions

Scalar valued smooth functions

Definition

A function $f : \mathbb{R}^n \to \mathbb{R}$ is smooth if all the partial derivatives,

$$\partial_{i_1 \cdots i_k} f := \partial_{x^{i_1}} \cdots \partial_{x^{i_k}} := \frac{\partial^k f}{\partial x^{i_1} \cdots \partial x^{i_k}}$$

exist and are continuous.

- Here $k \in \mathbb{N}$ and $1 \leq i_1, \ldots, i_k \leq n$ are any choice of k indices between 1 and n.
- For $k = 0$, there are no derivatives and in this case, the condition is just that f is continuous.
- The differential in the standard basis $\{e_i\}_{i=1}^n$ is now

$$df_x = \begin{pmatrix} \partial_1 f & \cdots & \partial_n f \end{pmatrix}.$$

- Now we have a function $x \in \mathbb{R}^n \mapsto df_x \in \mathbb{R}^n$. To define the second derivative, we need to differentiate functions $\mathbb{R}^n \to \mathbb{R}^n$.

Vector valued smooth functions

Definition

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is *smooth* if the component functions f_i, $1 \leq i \leq m$ are smooth where $f(x) = (f^1(x), \ldots, f^m(x)) \in \mathbb{R}^m$.

- Notice that for each $i = 1, \cdots, n$, the differential, df_i is a $1 \times n$ matrix. That is, the differential becomes an $m \times n$ matrix:

$$df_x = \begin{pmatrix} \partial_1 f^1 & \cdots & \partial_n f^1 \\ \vdots & \ddots & \vdots \\ \partial_1 f^m & \cdots & \partial_n f^m \end{pmatrix}.$$

Higher derivatives

- For $f : \mathbb{R}^n \to \mathbb{R}$ we now have

$$d^2 f = d(df) = \begin{pmatrix} \partial_{11} f & \cdots & \partial_{1n} f \\ \vdots & \ddots & \vdots \\ \partial_{n1} f & \cdots & \partial_{nn} f \end{pmatrix}$$

- For $f : \mathbb{R}^n \to \mathbb{R}^m$, write $(df_x)_{ij} = (\partial_i f^j)_{ij}$ for the differential and observe that differentiating again, gives for each component, $\partial_i f^j$

$$\partial_k \partial_i f^j.$$

- In other words, for each $1 \leq j \leq m$, we get a matrix $d^2 f^j$.
- These are *tensors*. In general $d^k f$ is an order $(k+1)$ object, indexed by indices $1 \leq j \leq m$ and $1 \leq i_1, \cdots, i_k \leq n$.

The graph of a function

Definition

Let $f : U \subseteq_{\text{open}} \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. The graph, $\text{Gr}\, f$ is the set,

$$\text{Gr}\, f := \{(u, v, f(u, v)) : (u, v) \in U\} \subseteq \mathbb{R}^3.$$

The function $F : U \to \mathbb{R}^3$ defined by

$$F(u, v) = (u, v, f(u, v))$$

is a *parmetrisation* of $\text{Gr}\, f$.
Notice that the function F is smooth and gives a one to one identification of the points $(x, y, z) \in \text{Gr}\, f$ with the points $(u, v) \in U$ an open set of $\mathbb{R}^2$ on which we can do calculus!

Smooth functions on a graph

Definition

A function $\varphi : \operatorname{Gr} f \to \mathbb{R}$ is smooth if the function

$$\varphi \circ F(x, y) = \varphi(x, y, f(x, y))$$

is smooth. A function $\varphi = (\varphi^1, \dots, \varphi^m) : \operatorname{Gr} f \to \mathbb{R}^m$ is smooth if each φ^i is.

If $\Phi : \mathbb{R}^3 \to \mathbb{R}$ is smooth then, by the chain rule $\varphi := \Phi|_{\operatorname{Gr} f}$ is smooth since

$$\varphi \circ F = \Phi|_{\operatorname{Gr} f} \circ F = \Phi \circ F$$

is the composition of smooth functions.

Extension of smooth functions on a graph

Lemma

Let $\varphi : \operatorname{Gr} f \to \mathbb{R}$ be a smooth function. Then locally there exists a smooth function $\Phi : \mathbb{R}^3 \to \mathbb{R}$ such that $\varphi = \Phi_{\operatorname{Gr} f}$.

Proof.

- Define $G(u, v, w) = (u, v, w + f(u, v))$ for $(u, v, w) \in U \times \mathbb{R}$.
- Then $G(u, v, 0) = F(u, v)$ parametrises $\operatorname{Gr} f$.
- The differential is nonsingular:

$$dG = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \partial_u f & \partial_v f & 1 \end{pmatrix}.$$

- Hence by the inverse function theorem, there is a neighbourhood of W of each $(u_0, v_0, 0)$ and a neighbourhood V of $(x_0, y_0, z_0) = (u_0, v_0, f(u_0, v_0))$ such that $G : W \to V$ is

Extension of smooth functions on a graph

Proof.

- $G = (u, v, w + f(u, v))$ is a local diffeomorphism around $(u_0, v_0, 0)$.
- Now define the smooth function

$$\Phi(x, y, z) = \varphi \circ \bar{F} \circ G^{-1}.$$

where $\bar{F}(u, v, w) = F(u, v) = G(u, v, 0)$.

- Note that Φ is defined on an open set of $\mathbb{R}^3$ and not just on $\operatorname{Gr} f$.
- Then for $(x, y, z) = F(u, v) \in \operatorname{Gr} f$, we have $G^{-1}(x, y, z) = (u, v, 0)$ and hence

$$\Phi|_{\operatorname{Gr} f}(x, y, z) = \varphi \circ \bar{F}(u, v, 0) = \varphi(u, v, f(u, v)) = \varphi(x, y, z).$$

□

Lecture Five: Surfaces that are graphs - The tangent plane to a graph

Tangent Vectors

- If $\varphi : \mathrm{Gr}\, f \to \mathbb{R}$ is smooth we know what $d(\varphi \circ F)$ is. But what is $d\varphi$?
- Indeed, as a linear map, what is the domain of $d\varphi$?

Definition

A *tangent vector* at x to $\mathrm{Gr}\, f$ is a vector $X \in \mathbb{R}^3$ such that there exists a curve $\gamma : (-\epsilon, \epsilon) \to \mathrm{Gr}\, f \subseteq \mathbb{R}^3$ with

$$\gamma(0) = x, \quad \gamma'(0) = X.$$

The *tangent plane*, $T_x \mathrm{Gr}\, f$ to $\mathrm{Gr}\, f$ at x is the set of tangent vectors at x.

- Tangent vectors are velocity vectors to curves *along the graph*.

Tangent plane

Lemma

The tangent plane, $T_x \operatorname{Gr} f = dF_{(u,v)}(\mathbb{R}^2)$ *is a plane in* $\mathbb{R}^3$ *where* $F(u,v) = x$.

Proof.

Let $(u,v) \in \mathbb{R}^2$ be the unique point such that $x = F(u,v)$.
We have

$$dF_{(u,v)}(\mathbb{R}^2) = \{c^1 dF \cdot e_1 + c^2 dF \cdot e_2 = dF(c^1 e_1 + c^2 e_2) : c^1, c^2 \in \mathbb{R}\}.$$

and

$$T_x \operatorname{Gr} f = \{\gamma'(0) : \gamma(0) = x\}.$$

Tangent plane: $df(\mathbb{R}^2) \subseteq T_x \operatorname{Gr} f$.

Proof.

Since $F : U \to \mathbb{R}^3$ with U open, given any c^1, c^2, there exists an $\epsilon > 0$ such that

$$\gamma(t) = (u, v) + t(c^1 e_1 + c^2 e_2) \in U, \quad t \in (-\epsilon, \epsilon).$$

Then $F \circ \gamma : (-\epsilon, \epsilon) \to \operatorname{Gr} f$ satisfies $F \circ \gamma(0) = x$ and

$$X = (F \circ \gamma)'(0) = dF_{(u,v)} \cdot \gamma'(0) = dF_{(u,v)} \cdot (c^1 e_1 + c^2 e_2) \in T_x \operatorname{Gr} f.$$

Thus $dF_{(u,v)}(\mathbb{R}^2) \subseteq T_x \operatorname{Gr} f$.

Tangent plane: $T_x \operatorname{Gr} f \subseteq df(\mathbb{R}^2)$.

Proof.

Let $X = \gamma'(0)$.
Define

$$\mu(t) = \pi \circ G^{-1} \circ \gamma(t)$$

where $\pi : (u, v, w) = (u, v)$ is orthogonal projection onto the (u, v) plane.
Recall that if $\gamma(t) = (x(t), y(t), z(t)) \in \operatorname{Gr} f$, then

$$G^{-1}(x(t), y(t), z(t)) = (u(t), v(t), 0).$$

with $F(u(t), v(t)) = (x(t), y(t), z(t))$.
Thus letting $\mu'(0) = c^1 e_1 + c^2 e_2$ we have

$$dF_{(u,v)}(c^1 e_1 + c^2 e_2) = (F \circ \mu)'(0) = (F \circ \pi \circ G^{-1} \circ \gamma)'(0) = \gamma'(0) = X.$$

Thus $T_x \operatorname{Gr} f \subseteq dF_{(u,v)}(\mathbb{R}^2)$. □

Vector space structure on the tangent plane

We have two ways of realising the tangent plane as a vector space:

1. $T_x \operatorname{Gr} f$ is a set of vectors in $\mathbb{R}^3$. So they inherit a vector space structure directly from $\mathbb{R}^3$!
   In terms of curves, let $X_i = \gamma_i'(0)$ with $\gamma_i(0) = x$ for $i = 1, 2$.
   Note that if $\gamma_i(t) = (x_i(t), y_i(t), z_i(t))$ then $\gamma_i'(0) = (x_i'(0), y_i'(0), z_i'(0))$. The vector space operations are then
   $$c^1 X_1 + c^2 X_2 = \mu'(0)$$
   where
   $$\mu(t) = x + c^1(\gamma_1(t) - x) + c^2(\gamma_2(t) - x)$$
   Then $\mu'(0) = c^1\gamma_1'(0) + c^2\gamma_2'(t) = c^1 X_1 + c^2 X_2$.
2. $\mathbb{R}^2$ is already a vector space and dF is injective since
   $$F_u = dF(e_1) = \partial_u F = e_1 + \partial_u f, \quad F_v = dF(e_2) = \partial_v F = e_2 + \partial_v f$$
   are linearly independent.Then $c^1 X_1 + c^2 X_2 = dF(c^1 Y_1 + c^2 Y_2)$ where $dF(Y_i) = X_i$ with Y_i uniquely determined.

Vector space structure on the tangent plane

Exercise: Show that the map

$$A : c^1 e_1 + c^2 e_2 \mapsto \partial_t|_{t=0} F((u,v) + t(c^1 e_1 + c^2 e_2))$$

induces a linear isomorphism between $dF_{(u,v)}(\mathbb{R}^2)$ and $T_x \operatorname{Gr} f$.
Thus the two vector space structures are equivalent in the sense that they are isomorphic.

The differential

Now let $\varphi : \mathrm{Gr}\, f \to \mathbb{R}$. Then we have two ways to define

$$d\varphi : T_x \,\mathrm{Gr}\, f \to \mathbb{R}.$$

1. $$d\varphi(c^1 F_u + c^2 F_v) = d(\varphi \circ F)(c^1 e_1 + c^2 e_2).$$

2. $$d\varphi(c^1 X_1 + c^2 X_2) = \partial_t|_{t=0} \Phi\left(x + (\gamma_1(t) - x) + (\gamma_2(t) - x)\right).$$

   where Φ is any extension of φ. Why do we need to this? Does the result depend on the extension?

Exercise: Show that if $X = A(Y)$ from the isomorphism above, then $d\varphi Y = d\varphi X$ where the first $d\varphi$ is from the first definition and the second $d\varphi$ uses the second definition.