MATH704 Differential Geometry

Macquarie University, Semester 2 2018

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Lecture Eight: Geometry And Curvature Of Regular Surfaces

Lecture Eight: Geometry And Curvature Of Regular Surfaces - Geometry Of Surfaces

Length and Angle of Tangent Vectors

Definition

Let X be a tangent vector. Then it's length is defined to be

$$|X|_g := \sqrt{g(X,X)}.$$

Definition

The angle, θ between two tangent vectors X, Y (at the same point $x \in S$!) is defined by

$$\cos\theta = \frac{g(X,Y)}{|X|\,|Y|} = g\left(\frac{X}{|X|}, \frac{Y}{|Y|}\right).$$

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Cauchy Schwartz Inequality

Lemma

$$|g(X, Y)| \leq |X| \, |Y| \, .$$

See https://en.wikipedia.org/wiki/Cauchy%E2%80%93Schwarz_inequality#First_proof

Rearranging Cauchy-Schwarz inequality for $X, Y \neq 0$ gives

$$\frac{g(X, Y)}{|X| \, |Y|} \in [-1, 1]$$

and θ is well defined after choosing an inverse arccos.
The simplest is to take $\theta \in [0, \pi]$.

Arc Length

Definition

Let $\gamma : (a, b) \to S$ be a smooth curve. The *arc-length* of γ is

$$L(\gamma) = \int_a^b \left|\gamma'(t)\right| dt.$$

As for plane and space curves, define the arc length parameter

$$s(t) = \int_a^t \left|\gamma'(\tau)\right| d\tau$$

so that $s'(t) = |\gamma'(t)|$ is smoothly invertible for *regular* curves (i.e. with $\gamma'(t) \neq 0$).
Then we may parametrisse γ by arclength:

$$\gamma(s) = \gamma(t(s))$$

satisfying $|\gamma'| \equiv 1$.

Area

Let

$$X_u = d\varphi(e_u) = \partial_u \varphi, \quad X_v = d\varphi(e_v) = \partial_v \varphi$$

be coordinate vectors.
Since $d\varphi$ is injective, X_u, X_v form a basis for $T_x M$.
In fact X_u, X_v determines a parallelogram $X_u \wedge X_v \subseteq T_x M$.
Taking a small rectangle $R = \{(u, v) \in (u_0, u_0 + \Delta u) \times (v_0, v_0 + \Delta v)\}$, we approximate the area of $\varphi(R) \subseteq S$ by

$$\text{Area}(\varphi(S)) \simeq \text{Area}(X_u \wedge X_v) = |X_u \times X_v| \,\text{Area}(R) = |X_u \times X_v| \,\Delta u \Delta v.$$

Note that $|X_u \times X_v|^2 = \det \lambda^T \lambda = \det g$ where $\lambda = (X_u \quad X_v)$!
Area is the limit of a Riemann sum: for any region $\Omega = \varphi(W) \subseteq \varphi(U)$

$$\text{Area}(\Omega) = \int_W \sqrt{\det g(u, v)} du dv.$$

Intrinsic Geometry

- Notice that thinking of $\gamma : (a, b) \to \mathbb{R}^3$ we have

$$g(\gamma'(0), \gamma'(0)) = \langle \gamma'(0), \gamma'(0) \rangle_{\mathbb{R}^3}$$

  so that the length of tangent vectors and hence the length of curves is precisely the lengths obtained in $\mathbb{R}^3$.
- Similar for angles and for area in terms of X_u, X_v.
- The point is that, if we know g, we may do geometry on S without any reference to how it sits in $\mathbb{R}^3$! This is *intrinsic geometry*.
- But what exactly is the definition of g if we don't refer to $\mathbb{R}^3$?

At this point, the best we can do is say that g is determined by a collection of smooth, matrix valued maps $(u, v) \in U \mapsto (g_{ij}(u, v))$ in each local parametrisation that are symmetric and positive definite at each point (u, v). We also require that under a change of coordinates, τ we have

$$g^{\varphi\circ\tau}_{ab} = \sum_{ij} g^{\varphi}_{ij} \partial_{y^a}\tau^i \partial_{y^b}\tau^j.$$

Lecture Eight: Geometry And Curvature Of Regular Surfaces - Orientation And The Gauss Map

Orientation of Euclidean Space

Definition

An orientation on $\mathbb{R}^n$ is an equivalence class of *ordered* bases $\mathcal{E} = (e_1, \cdots, e_n)$ where $\mathcal{E} \sim \mathcal{F}$ if the change of basis matrix $A_{\mathcal{E}\mathcal{F}}$ has positive determinant.

Since $\det(A_{\mathcal{E}\mathcal{F}} A_{\mathcal{F}\mathcal{G}}) = \det(A_{\mathcal{E}\mathcal{F}}) \det(A_{\mathcal{F}\mathcal{G}})$, we do indeed have an equivalence relation, and there are *precisely two equivalence classes*.

Example

Compute the change of basis from $\mathcal{E} = (e_1, e_2)$ to
$(e_1, e_1 + e_2), \quad (e_1, -e_2), \quad (e_2, e_1)$.

Example

Right hand orientation: $(e_1, e_2, e_3), (e_1, e_3, -e_2), \ldots$
Left hand orientation: $(e_2, e_1, e_3), (e_1, -e_2, e_3), \ldots$

Orientation preserving and reversing linear maps

Choose an orientation $\mathcal{O} = \{e_1, \cdots, e_n\}$ on $\mathbb{R}^n$.

Definition

An *invertible* linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ is orientation preserving if $T(\mathcal{O}) = \mathcal{O}$. That is, if

$$\det\big(T(e_1), \cdots, T(e_n)\big) = \det\big(e_1, \cdots, e_n\big)$$

or equivalently if $\det T > 0$.

Example

$$\text{Preserving:} \quad T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}.$$

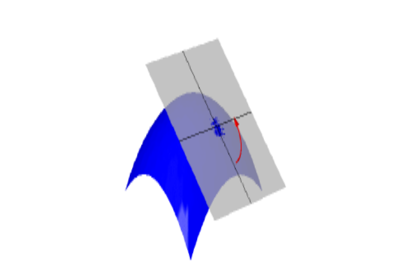
$$\text{Reversing:} \quad T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}.$$

Orientation of the tangent plane

Tangent Plane Orientations

Local parametrisation: $\varphi : U \to S$.

$$\left(\frac{\partial\varphi}{\partial u}, \frac{\partial\varphi}{\partial v}\right), \quad \left(\frac{\partial\varphi}{\partial v}, \frac{\partial\varphi}{\partial u}\right)$$



Definition

The orientation induced by φ is *compatible* with the orientation induced by ψ if $\det d(\psi \circ \phi^{-1}) > 0$. A regular surface, S is *orientable* if there is a cover $\varphi_\alpha : U_\alpha \to S$ such that $\det(\tau_{\alpha\beta}) > 0$ for all α, β.

Examples

- The sphere is orientable
- The Möbius strip is *not* orientable
- Graphs, are orientable
- Inverse images of regular point are orientable: here $F : \mathbb{R}^3 \to \mathbb{R}$, $S = F^{-1}(0)$ where dF_x has maximal rank (i.e. rank 1) for all $p \in \mathbb{R}^3$ such that $F(p) = 0$.

Orientation of surfaces

Theorem

A surface S is orientable if and only if there is a differentiable field, N of unit normal vectors. That is, if and only there exists a differentiable map $N : S \to \mathbb{R}^3$ such that $|N(x)| = 1$ for all $x \in S$ and such that $N(x) \perp X$ for all $X \in T_xS$.

Remember there are precisely two orientations!

There are two possible unit normal fields, N and $-N$. Choosing an orientation is equivalent to choosing a normal field.

- The proof of the theorem follows from the following lemma:

Lemma

Let $\varphi(u, v) : U \subseteq \mathbb{R}^2 \to S$ and $\psi(s, t) : V \subseteq \mathbb{R}^2 \to S$ be local parametrisations. Then

$$\partial_u\varphi \times \partial_v\varphi = \left[\det d(\psi^{-1} \circ \varphi)\right] \partial_s\psi \times \partial_t\psi.$$

Gauss Map

Definition

An orientable surface S along with a choice of orientation is called an *oriented surface*.

Definition

Let S be an oriented surface. The *Gauss Map* is the unit normal map

$$x \in S \mapsto N(x) \in \mathbb{S}^2 = \{X \in \mathbb{R}^3 : \|X\| = 1\}.$$

With respect to a local parametrisation

$$N = \frac{\partial_u \varphi \times \partial_v \varphi}{|\partial_u \varphi \times \partial_v \varphi|}.$$

Examples

Sphere:

$$S = \{x^2 + y^2 + z^2 = 1\}, \quad N(p) = p$$

Graph:

$$S = \{(x, y, f(x, y))\}, \quad N(x, y, f(x)) = \frac{1}{\sqrt{1 + f_x^2 + f_y^2}}(-f_x, -f_y, 1).$$

Inverse image of regular point

$$S = \{F^{-1}(c)\}, \quad N(p) = \frac{\nabla F(p)}{|\nabla F(p)|}.$$

Lecture Eight: Geometry And Curvature Of Regular Surfaces - Curvature

Weingarten Shape Operator

Definition

The *Weingarten* or *Shape* Operator at $p \in S$ is the linear map

$$\mathcal{W} = -dN_p : T_pS \to T_pS.$$

Note that $N : S \to \mathbb{S}^2$ so that $dN_p : T_pS \to T_{N(p)}\mathbb{S}^2$. By definition

$$N(p) \perp T_pS$$

But on the sphere, $N_{\mathbb{S}^2}(z) = z$ and hence (with $z = N(p)$)

$$N(p) \perp T_{N(p)}\mathbb{S}^2.$$

Therefore, $T_{N(p)}\mathbb{S}^2$ is a plane parallel to T_pS so we may *identify* these planes to obtain $dN_p : T_pS \to T_{N(p))}\mathbb{S}^2 \simeq T_pS$.

Examples

Plane

$$S = \{ax + by + cz = 0\}$$

$$N(p) = (a, b, c)$$

$$dN_p \equiv 0$$

Sphere

$$S = \mathbb{S}^2 = \{x^2 + y^2 + z^2 = 1\}$$

$$N(p) = p$$

$$dN_p = \mathrm{Id}\,.$$

Examples

Cylinder

$$C = \{x^2 + y^2 = 1, -1 < z < 1\}$$

$$N(p) = \pi_{(x,y)}(p) : N(x,y,z) = (x,y,0).$$

$$dN_p = \pi_{x,y}$$

Tangent vectors at $p = (\cos\theta, \sin\theta, z_0)$:

$$X = (-\sin\theta, \cos\theta, 0), \quad Y = (0,0,1)$$

$$dN_p X = \left.\frac{d}{dt}\right|_{t=0} N(\cos(\theta+t), \sin(\theta+t), z) = X$$

$$dN_p Y = \left.\frac{d}{dt}\right|_{t=0} N(\cos\theta, \sin\theta, z+t) = 0.$$

Interpretation of $\mathcal{W}$

Curvature of a plane curve

$$\kappa = \langle \partial_s T, N \rangle = - \langle T, \partial_s N \rangle = -dN(T).$$

Measures the change of T, or equivalently, N along the curve.

Curvature of a surface

- For surfaces T_pS is *two-dimensional*.
- $\mathcal{W}(V) = -dN(V)$ measures change of N in the direction V:
  Let γ be a curve with $\gamma(0) = p$, and $V = \gamma'(0)$. Then

$$dN_p(V) = \partial_t|_{t=0} N(\gamma(t)) = \text{ deviation of } N \text{ along the curve } \gamma.$$

- Thus dN is measures how the surface is curved in two-dimensions.