MATH704 Differential Geometry

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Lecture Nine: Curvature Of Regular Surfaces

Lecture Nine: Curvature Of Regular Surfaces - Geodesic and Normal Curvature

Geodesic and normal Curvature

- Let $\gamma : I \to S$ be a curve on S, $p = \gamma(0)$, $V = \gamma'(0) \in T_pS$.
- Note: The normal (in $\mathbb{R}^3$) $n_{\mathbb{R}^3}$ to γ may be tangent to S, or may be normal to S, or some linear combination thereof.
- As a curve in $\mathbb{R}^3$, γ may have curvature, $\kappa_{\mathbb{R}^3} \neq 0$ simply because S has curvature!

Definition

The *normal curvature* of γ is the part of the curvature normal to S:

$$\kappa_N = \langle \kappa n_{\mathbb{R}^3}, N \rangle .$$

The geodesic curvature vector, $\overrightarrow{\kappa}_S$ (along S) is the projection of the curvature vector $\overrightarrow{\kappa}_{\mathbb{R}^3} = \kappa_{\mathbb{R}^3} n_{\mathbb{R}^3}$ onto the tangent plane:

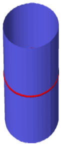
$$\overrightarrow{\kappa}_S = \pi_{T_pS}(\kappa_{\mathbb{R}^3} n_{\mathbb{R}^3}) = \kappa_{\mathbb{R}^3} n_{\mathbb{R}^3} - \langle \kappa_{\mathbb{R}^3} n_{\mathbb{R}^3}, N \rangle N.$$

Let $n_S \in T_pS$ be such that $n_S \perp \gamma'(0)$ and $(\gamma'(0), n_S)$ has positive orientation. The geodesic curvature is $\kappa_S = \langle \overrightarrow{\kappa}_S, n_S \rangle$ (has a sign!).

Example

Cylinder: $C = \{x^2 + y^2 = 1, -1 < z < 1\}, \quad N(x, y, z) = (x, y, 0).$

$$\begin{aligned}
\gamma(t) &= (\cos t, \sin t, z_0) \\
\gamma'(t) &= (-\sin t, \cos t, 0) \\
\gamma''(t) &= (-\cos t, -\sin, 0) \\
N(\gamma(t)) &= (\cos t, \sin t, 0) \\
n_S(\gamma(t)) &= (0, 0, 1) \\
\kappa_{\mathbb{R}^3} &= \kappa_N = 1, \kappa_S = 0.
\end{aligned}$$



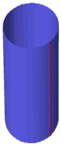
- Check the orientation!

$$(\gamma', n_S, N)$$

Example

Cylinder: $C = \{x^2 + y^2 = 1, -1 < z < 1\}, \quad N(x,y,z) = (x,y,0).$

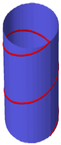
$$\begin{aligned}
\gamma(t) &= (1,0,t) \\
\gamma'(t) &= (0,0,1) \\
\gamma''(t) &= (0,0,0) \\
N(\gamma(t)) &= (1,0,0) \\
n_S(\gamma(t)) &= (0,-1,0) \\
\kappa_{\mathbb{R}^3} &= \kappa_N = \kappa_S = 0.
\end{aligned}$$



Example

Cylinder: $C = \{x^2 + y^2 = 1, -1 < z < 1\}, \quad N(x, y, z) = (x, y, 0).$

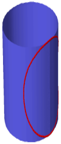
$$\begin{aligned}
\gamma(t) &= (\cos t, \sin t, t) \\
\gamma'(t) &= (-\sin t, \cos t, 1) \\
\gamma''(t) &= (-\cos t, -\sin, 0) \\
N(\gamma(t)) &= (\cos t, \sin t, 0) \\
n_S(\gamma(t)) &= \frac{1}{\sqrt{2}}(\sin t, -\cos t, 1) \\
\kappa_{\mathbb{R}^3} &= \kappa_N = 1, \kappa_S = 0.
\end{aligned}$$



Example

Cylinder: $C = \{x^2 + y^2 = 1, -1 < z < 1\}, \quad N(x, y, z) = (x, y, 0).$

$$\begin{aligned}
\gamma(t) &= (\cos(\cos t), \sin(\cos t), \sin t) \\
\gamma'(t) &= (\sin(\sin t)\cos t, -\cos(\cos t)\sin t, \cos t) \\
\gamma''(t) &= (-\cos(\cos t)\sin^2 t, -\sin(\cos t)\sin^2 t, -\sin t) \\
N(\gamma(t)) &= (\cos(\cos t), \sin(\cos t), 0) \\
n_S(\gamma(t)) &=? \\
\kappa_{\mathbb{R}^3} &= \kappa_N =?, \kappa_S = 1.
\end{aligned}$$



- How do I know $\kappa_S = 1$?
- You have to wait to find out!

Example

$$\kappa_{\mathbb{R}^3} = \frac{1}{\left(\left|\cos\left(\cos\left(t\right)\right)\sin\left(t\right)\right|^2 + \left|\sin\left(t\right)\sin\left(\cos\left(t\right)\right)\right|^2 + \left|\cos\left(t\right)\right|^2\right)^{\frac{3}{2}}} \times$$

$$\sqrt{\left|-\left(\cos\left(\cos\left(t\right)\right)\sin\left(t\right)^2 - \cos\left(t\right)\sin\left(\cos\left(t\right)\right)\right)\cos\left(\cos\left(t\right)\right)\sin\left(t\right)\right.}$$

$$\left.-\left(\sin\left(t\right)^2\sin\left(\cos\left(t\right)\right) + \cos\left(t\right)\cos\left(\cos\left(t\right)\right)\right)\sin\left(t\right)\sin\left(\cos\left(t\right)\right)\right|^2$$

$$+\left|\cos\left(\cos\left(t\right)\right)\sin\left(t\right)^2\right.$$

$$\left.+\left(\sin\left(t\right)^2\sin\left(\cos\left(t\right)\right) + \cos\left(t\right)\cos\left(\cos\left(t\right)\right)\right)\cos\left(t\right)\right|^2$$

$$+\left|\sin\left(t\right)^2\sin\left(\cos\left(t\right)\right)\right.$$

$$\left.-\left(\cos\left(\cos\left(t\right)\right)\sin\left(t\right)^2 - \cos\left(t\right)\sin\left(\cos\left(t\right)\right)\right)\cos\left(t\right)\right|^2$$

Lecture Nine: Curvature Of Regular Surfaces - Curvature and the Second Fundamental Form

Dependence of normal curvature on direction

Theorem

Let $\gamma, \sigma : I \to S$ *with* $\gamma(t_0) = \sigma(t_0)$ *and* $\gamma'(t_0) = \sigma'(t_0)$ *for some* $t_0 \in I$. *Then*

$$\kappa_N(\gamma)(t_0) = \kappa_N(\sigma)(t_0).$$

That is, the normal curvature κ_N *depends only the tangent vector* $V = \gamma'(t_0) = \sigma'(t_0)$ *at the point* $p = \gamma(t_0) = \sigma(t_0)$.

Note: Both $\kappa_{\mathbb{R}^3}$ and κ_S also depend on $\gamma''(t_0)$ (resp. $\sigma''(t_0)$) and so *will in general differ* for γ and σ if $\gamma''(t_0) \neq \sigma''(t_0)$.

Thus κ_N measures the curvature of S itself in the direction V independently of the choice of curve γ, σ.

Whereas κ_S measures the "left-over" curvature of γ after removing the curvature of S itself.

Proof of Theorem

We will show that

$$\kappa_N = -\left\langle dN(\gamma'), \gamma' \right\rangle .$$

Let γ be parametrised by arc-length, s. Then

$$\kappa_{\mathbb{R}^3} n_{\mathbb{R}^3} = \gamma''.$$

Therefore,

$$\kappa_N = \langle \kappa_{\mathbb{R}^3} n_{\mathbb{R}^3}, N \rangle = \left\langle \gamma'', N \right\rangle .$$

On the other hand, since $\langle \gamma', N \rangle = 0$ we have

$$0 = \partial_s \left\langle \gamma', N \right\rangle = \left\langle \gamma'', N \right\rangle + \left\langle \gamma', dN(\gamma') \right\rangle .$$

Hence,

$$\kappa_N = \left\langle \gamma'', N \right\rangle = -\left\langle \gamma', dN(\gamma') \right\rangle .$$

The Second Fundamental Form

Definition

The *second fundamental form*, or *extrinsic curvature* of S is defined to be

$$A(X, Y) = g(\mathcal{W}(X), Y) = \langle \mathcal{W}(X), Y \rangle = \langle -dN(X), Y \rangle$$

for X, Y tangent vectors to S.

Classically, the second fundamental form is the *quadratic form*:

$$\mathrm{II}(\mathrm{X}) = \mathrm{A}(\mathrm{X}, \mathrm{X}).$$

The theorem shows that for any curve γ on S parametrised by arc length,

$$\kappa_N = \left\langle -dN(\gamma'), \gamma' \right\rangle = A(\gamma', \gamma') = \mathrm{II}(\gamma').$$

More generally

$$\kappa_N = \frac{\mathrm{II}(\gamma')}{|\gamma'|^2} = \frac{A(\gamma', \gamma')}{g(\gamma', \gamma')} = A\left(\frac{\gamma'}{|\gamma'|}, \frac{\gamma'}{|\gamma'|}\right).$$

Spheres

Radius 1: $\mathbb{S}^2 = \{x^2 + y^2 + z^2 = 1\}$

Choose $N(p) = -p$ (inward pointing). Then $dN_p(X) = -X$ and

$$A(X, Y) = \langle -dN(X), Y \rangle = \langle X, Y \rangle = g(X, Y)$$

Radius r: $\mathbb{S}^2(r) = \{x^2 + y^2 + z^2 = r^2\}$

Choose $N(p) = -\frac{1}{r}p$ Then $dN_p(X) = -\frac{1}{r}X$ and

$$A(X, Y) = \langle -dN(X), Y \rangle = \frac{1}{r}\langle X, Y \rangle = \frac{1}{r}g(X, Y)$$

Equators (Great Circles)

$\gamma(\theta) = (r\cos(\theta), r\sin(\theta), 0)$: $\kappa_{\mathbb{R}^3} = \kappa_N = \frac{1}{r}$.
$A(\gamma', \gamma') = \frac{1}{r}g(\gamma', \gamma') = \frac{1}{r}r^2 = r \neq \kappa_N$?????
$A(\gamma', \gamma') = |\gamma'|^2\kappa_N$ - not arc-length!

Symmetry

Theorem

The second fundamental form is symmetric: $A(X, Y) = A(Y, X)$*.*
Equivalently, the Weingarten shape operator is self-adjoint with respect to g*:* $g(\mathcal{W}(X), Y) = g(X, \mathcal{W}(Y))$*.*

Proof.

Recall $A(X, Y) = -\langle dN(X), Y\rangle$.
Let $\gamma(s) \in S$ be a curve with $\gamma'(0) = X$.
Then since $\langle N(\gamma(s)), Y(\gamma(s))\rangle = 0$ we have,

$$\begin{aligned} 0 &= \partial_s \langle N(\gamma(s)), Y(\gamma(s))\rangle \\ &= \left\langle dN(\gamma'), Y\right\rangle + \left\langle N, dY(\gamma')\right\rangle \\ &= \langle dN(X), Y\rangle + \langle N, dY(X)\rangle . \end{aligned}$$

Likewise $0 = \langle dN(Y), X\rangle + \langle N, dX(Y)\rangle$.

Symmetry (proof continued)

Proof.

Thus $A(X, Y) = -\langle dN(X), Y\rangle = \langle N, dY(X)\rangle$ and
$A(Y, X) = \langle N, dX(Y)\rangle$.
The required result is equivalent to the statement that $dX(Y) - dY(X)$ is tangential, since then

$$A(X, Y) - A(Y, X) = \langle N, dY(X) - dX(Y)\rangle = 0.$$

Let's take the case, $X = \partial_u\varphi$, $Y = \partial_v\varphi$ in a local parametrisation φ:
In this case,

$$\langle N, dX(Y)\rangle = \langle N, \partial_v\partial_u\varphi\rangle = \langle N, \partial_u\partial_v\varphi\rangle = \langle N, dY(X)\rangle .$$

The general result follows by bi-linearity of A and that $\{\partial_u\varphi, \partial_v\varphi\}$ is a basis so any X, Y are linear combinations of them. exercise! □

Lecture Nine: Curvature Of Regular Surfaces - Principal, Mean and Gauss Curvatures

Principal curvatures and Principal Directions

Definition

The *principal curvatures* κ_1, κ_2 are the eigenvalues of the Weingarten shape operator. The eigenvectors, e_1, e_2 are called *principal directions*.

- Note that the principal curvatures (and directions) vary from point to point, since dN varies from point to point.
- From above, we know that dN is self-adjoint.
- From the appendix below we know that dN (being self adjoint) has an orthonormal basis e_1, e_2 of eigenvectors with eigenvalues κ_1, κ_2.
- With respect to e_1, e_2,

$$dN = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}.$$

Examples

Example

The sphere $\mathbb{S}^2(r) = \{x^2 + y^2 + z^2 = r^2\}$.

- $dN = -\frac{1}{r}\,\mathrm{Id}$: $\kappa_1 = \kappa_2 = \frac{1}{r}$.
- All directions are principal directions!

Example

The cylinder $\mathbb{C}^2(r) = \{x^2 + y^2 = r^2\}$.

- $dN = -\frac{1}{r}\pi_{\{z=0\}}$
- In the local parametrisation $(r\cos\theta, r\sin\theta, z)$:

$$dN = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{r} \end{pmatrix}.$$

- $\kappa_1(r,\theta) = 0, \quad e_1(r,\theta) = (0,0,1)$.
- $\kappa_2(r,\theta) = \frac{1}{r}, \quad e_2(r,\theta) = (-\sin\theta, \cos\theta, 0)$

Mean Curvature and Gauss Curvature

Definition

The *Mean Curvature* is

$$H := \mathrm{Tr}(\mathcal{W}) = \mathrm{Tr}(-dN) = \frac{1}{2}\left(\kappa_1 + \kappa_2\right).$$

The *Gauss Curvature* is

$$K := \det(\mathcal{W}) = \det(-dN) = \kappa_1 \kappa_2.$$

Examples

Plane $\mathbb{R}^2$

- $H = 0$
- $K = 0.$

Sphere $\mathbb{S}^2$

- $H = \frac{1}{r}$
- $K = \frac{1}{r^2}.$

Cylinder $\mathbb{C}^2(r)$

- $H = \frac{1}{2r}$
- $K = 0.$

Umbilic Points

Theorem

A point $p \in S$ is called umbilic if $\kappa_1 = \kappa_2$. If every point of a connected regular surface S is umbilic, then S is entirely contained in a plane, or a sphere.

- At an umbilic point p,
$$dN_p = \kappa(p)\,\mathrm{Id}$$
where $\kappa_1(p) = \kappa_2(p) = \kappa(p)$.
- Therefore, umbilic points are points where the surface curves the same way in all directions.
- The basic idea is to show that $\kappa(p) \equiv$ constant.

Proof of Umbilic Point Theorem: $\kappa \equiv$ constant.

- With respect to a local parametrisation with $\varphi_u = \partial_u \varphi, \varphi_v = \partial_v \varphi$:

$$dN(\varphi_u) = \partial_u N, \quad dN(\varphi_v) = \partial_v N.$$

- Thus $dN = \kappa \, \mathrm{Id}$ gives,

$$\partial_u N = \kappa \varphi_u, \quad \partial_v N = \kappa \varphi_v.$$

- What's next? Differentiate!

$$\partial_v \partial_u N = \kappa_v \varphi_u + \kappa \partial_v \partial_u \varphi$$

and

$$\partial_u \partial_v N = \kappa_u \varphi_v + \kappa \partial_u \partial_v \varphi$$

- Subtracting and use Claireaut's Theorem for mixed partial derivatives:

$$\kappa_v \varphi_u = \kappa_u \varphi_v \Rightarrow \kappa_v = \kappa_u = 0 \Rightarrow \kappa \equiv \text{constant}$$

since φ_u, φ_v are linearly independent.

Proof of Umbilic Point Theorem: Locally $S \subseteq \mathbb{R}^2$

- If we have
$$dN \equiv 0.$$

- Therefore
$$\partial_u \langle \varphi, N \rangle \underset{\text{prod rule}}{=} \langle \varphi_u, N \rangle + \langle \varphi, dN(\varphi_u) \rangle \underset{dN \equiv 0}{=} \langle \varphi_u, N \rangle \underset{\varphi_u \text{ tang}}{=} 0$$

  Likewise
$$\partial_v \langle \varphi, N \rangle = 0.$$

  Therefore $\langle \varphi, N \rangle =$ constant and the points $\varphi(u, v)$ lie in a plane.

Proof of Umbilic Point Theorem: Locally $S \subseteq \mathbb{S}^2$.

- If we have

$$dN = \kappa \, \mathrm{Id}, \quad \kappa \neq 0$$

- Therefore

$$\partial_u \left(\varphi - \tfrac{1}{\kappa} N\right) \underset{\kappa \equiv \text{const}}{=} \varphi_u - \frac{1}{\kappa} dN(\varphi_u) \underset{dN = \kappa \, \mathrm{Id}}{=} \varphi_u - \frac{1}{\kappa} \kappa \varphi_u = 0.$$

- Likewise

$$\partial_v \left(\varphi - \tfrac{1}{\kappa} N\right) = 0.$$

- Therefore

$$\varphi - \frac{1}{\kappa} N = y_0 \in \mathbb{R}^3 \text{ is constant.}$$

- and hence

$$|\varphi(u, v) - y_0| = \frac{1}{|\kappa|} \Rightarrow \varphi(u, v) \in \mathbb{S}^2\left(\frac{1}{|\kappa|}, y_0\right).$$

Proof of Umbilic Theorem: Global

- The local theorem establishes, for each local parametrisation φ:

$$\kappa_\varphi \equiv \text{constant}$$

$$\begin{cases} N_\varphi \equiv \text{const}, \langle \varphi, N_\varphi \rangle \equiv C_\varphi, & \kappa_\varphi = 0 \Rightarrow S_\varphi \subseteq \mathbb{R}^2(N_\varphi, C_\varphi) \\ \varphi - \frac{1}{\kappa_\varphi} \equiv y_\varphi, & \kappa_\varphi \neq 0 \Rightarrow S_\varphi \subseteq \mathbb{S}^2(\frac{1}{|\kappa_\varphi|}, y_\varphi) \end{cases}$$

- In any overlap of charts, $U_\alpha \cap U_\beta$ all the constants must agree.
- S connected, means for any two points $p, q \in S$ there is a continuous path $\gamma : [0,1] \to S$ such that $\gamma(0) = p$, $\gamma(1) = q$.
- Cover the image $\gamma([0,1])$ by local parametrisations $\varphi_\alpha(U_\alpha)$ which gives a cover of $[0,1]$:

$$\varphi_\alpha^{-1}(U_\alpha)$$

- $[0,1]$ is *compact* so there is a finite cover $\{\varphi_i\}_{i=1}^n$. with $p \in \varphi_1(U_1)$, $q \in \varphi_n(U_n)$, $U_i \cap U_{i+1} \neq \emptyset$
- Thus $\kappa(p) = \kappa_{\varphi_1} = \kappa_{\varphi_2} = \cdots = \kappa_{\varphi_n} = \kappa(q)$. Similar for the other constants so the plane (or sphere) is globally defined. □

Lecture Nine: Curvature Of Regular Surfaces - Appendix: Symmetric bilinear forms

Symmetric Bilinear Forms

Definition

Let V be a real, finite dimensional vector space.
e.g. $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \cdots, \mathbb{R}^n, \dots$
A *bilinear form* B on V is a map

$$B : V \times V \to \mathbb{R}$$

such that for all $c_1, c_2 \in \mathbb{R}$ and $X_1, X_2, Y \in V$:

$$B(c_1X_1 + c_2X_2, Y) = c_1B(X_1, Y) + c_2B(X_2, Y)$$

and

$$B(Y, c_1X_1 + c_2X_2) = c_1B(Y, X_1) + c_2B(Y, X_2)$$

B is *symmetric* if for every $X, Y \in V$:

$$B(X, Y) = B(Y, X).$$

Inner Products

Definition

An *inner-product* g is a *positive-definite* bilinear form. That is, g is a bilinear form such that for all $X \in V$,

$$g(X, X) \geq 0, \quad g(X, X) = 0 \Rightarrow X = 0.$$

Example

Let $V = \mathbb{R}^2$, $X = (x_1, x_2)$, and $Y = (y_1, y_2)$. Define the *standard* inner product:

$$g(X, Y) = \langle X, Y \rangle := x_1 y_1 + x_2 y_2.$$

Inner Products

Example

Let A be any positive-definite, symmetric matrix. Define

$$g(X, Y) := X^T AY = \langle X, AY \rangle .$$

For example, let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.$$

Then

$$g(X, Y) = 2x_1y_1 + x_1y_2 + x_2y_1 + 3x_2y_2.$$

Note

$$g(X, X) = 2x_1^2 + 2x_1x_2 + 3x_2^2 = (x_1 + x_2)^2 + x_1^2 + 2x_2^2 \geq 0$$

with equality if and only if $X = (0, 0)$.

Canonical Isomorphism

Lemma

Let g be an inner-product on V. Then g induces a linear isomorphism between the vector space $\text{Hom}(V)$ *of linear transformations $V \to V$ and the vector space $B^2(V)$ of bilinear forms on V.*

The vector space structure on $\text{Hom}(V)$ is given by letting for each $X \in V$,

$$(c_1 T_1 + c_2 T_2)(X) := c_1 T_1(X) + c_2 T_2(X).$$

That is, given real constants $c_1, c_2 \in \mathbb{R}$ and linear transformations $T_1, T_2 : V \to V$, we define the linear transformation $c_1 T_1 + c_2 T_2$ by specifying it's value for each $X \in V$.
On the right hand side, note that $T_1(X) \in V$ so $c_1(T_1(X))$ is scalar multiplication using the vector space structure on V. Likewise for $c_2(T_2(X))$. The sum $(c_1 T_1(X)) + (c_2 T_2(X))$ is vector addition in V.
The vector space structure on linear maps $V \to V$ is defined *pointwise*.
Exercise: Figure out the vector space structure on bilinear forms. *Hint*: It's also defined pointwise.

Canonical Isomorphism

Proof.

Choose any basis $\{e_1, \ldots, e_n\}$ for V, and write for $X \in V$:

$$X = X^1 e_1 + \cdots + X^n e_n.$$

A basis for $\mathrm{Hom}(V)$ is given by the linear transformations

$$T^i_j(X^1 e_1 + \cdots X^n e_n) = X^i e_j.$$

A basis for $B^2(V)$ is given by the bilinear forms

$$B^{ij}(X^1 e_1 + \cdots X^n e_n, Y^1 e_1 + \cdots + Y^n e_n) = X^i Y^j.$$

Thus,

$$\dim \mathrm{Hom}(V) = \dim B^2(V) = n^2.$$

Thus they are isomorphic, being vector spaces of the same dimension, but something else is needed to get a *canonical* isomorphism...

Canonical Isomorphism

Example

Let $e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$ be the standard basis for $\mathbb{R}^n$.
The isomorphism induced by the standard inner-product defined on basis elements by $T^i_j \mapsto B^{ij}$ and then extended by linearity.

Proof.

In general, given $T \in \text{Hom}(V)$ define

$$[B_g(T)](X, Y) = g(T(X), Y).$$

Then $B_g(T)$ is a bilinear form.
The map

$$B_g : T \mapsto B_g(T)$$

is our desired isomorphism.
exercise: Verify linearity of the map B_g.

Canonical Isomorphism

Proof.

That B_g is an isomorphism follows since if $B_g(T_1) = B_g(T_2)$, then for every $X, Y \in V$:

$$0 = (B_g(T_1) - B_g(T_2))(X, Y) = g(T_1(X) - T_2(X), Y).$$

In particular for $Y = T_1(X) - T_2(X)$ we get that for every X

$$0 = g(T_1(X) - T_2(X), T_1(X) - T_2(X)) \Rightarrow T_1(X) - T_2(X) = 0$$

since g is positive-definite.
Therefore, $B_g(T_1) = B_g(T_2) \Rightarrow T_1 = T_2$ and the map B_g is *injective*.
Since $\dim \operatorname{Hom}(V) = \dim B^2(V) = n^2 < \infty$, the map is also surjective and hence an isomorphism. □

Self-adjoint operators

Let g be an inner-product on a finite-dimensional vector space V.

Definition

A *self-adjoint operator* (with respect to g) is an linear map $T : V \to V$ such that for every $X, Y \in V$

$$g(T(X), Y) = g(X, T(Y))$$

Lemma

A linear map $T : V \to V$ is self-adjoint if and only if $B_g(T)$ is a symmetric bilinear form.

Proof.

exercise: Back of the envelope calculation directly using the definitions. □

Eigenvalues and Eigenvectors

Theorem

A self adjoint operator T is diagonalisable. That is, there is a basis $\{e_i\}_{i=1}^n$ of eigenvalues.

Proof.

Consider the case $\dim V = 2$.

Let us write

$$|X|_g = \sqrt{g(X,X)}.$$

From a basis $\{X_1, X_2\}$ Gram-Schmidt gives an orthonormal basis:

$$\tilde{e}_1 = \frac{X_1}{|X_1|_g}, \tilde{e}_2 = \frac{X_2 - g(X_2, \tilde{e}_1)\tilde{e}_1}{|X_2 - g(X_2, \tilde{e}_1)\tilde{e}_1|_g}.$$

$$g(\tilde{e}_i, \tilde{e}_j) = \delta_{ij} := \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

Eigenvalues and Eigenvectors

Proof.

We may thus write

$$\mathbb{S}^1 = \{X : g(X,X) = 1\} = \{\cos\theta\tilde{e}_1 + \sin\theta\tilde{e}_2 : \theta \in [0,2\pi]\}.$$

Let $\lambda_1 = \min\{g(T(X),X) : X \in \mathbb{S}^1\}$.
The map

$$\theta \in [0,2\pi] \mapsto g(T(\cos\theta\tilde{e}_1 + \sin\theta\tilde{e}_2), \cos\theta\tilde{e}_1 + \sin\theta\tilde{e}_2)$$

is continuous hence there exists a $\theta_0 \in [0,2\pi]$ such that

$$\lambda_1 = g(T(\cos\theta_0\tilde{e}_1 + \sin\theta_0\tilde{e}_2), \cos\theta_0\tilde{e}_1 + \sin\theta_0\tilde{e}_2).$$

Our desired basis is the orthonormal (why?) pair:

$$\begin{aligned} e_1 &= \cos\theta_0\tilde{e}_1 + \sin\theta_0\tilde{e}_2 \\ e_2 &= -\sin\theta_0\tilde{e}_1 + \cos\theta_0\tilde{e}_2 \end{aligned}$$

Eigenvalues and Eigenvectors

Proof.

Let us write

$$E_1(\theta) = \cos\theta \tilde{e}_1 + \sin\theta \tilde{e}_2, \quad E_2(\theta) = -\sin\theta \tilde{e}_1 + \cos\theta \tilde{e}_2$$

so that

$$E_1(\theta_0) = e_1, \quad E_2(\theta_0) = e_2$$

and

$$E_1'(\theta) = E_2(\theta), \quad E_2'(\theta) = -E_1(\theta)$$

Eigenvalues and Eigenvectors

Proof.

By definition, θ_0 is a critical point of

$$g(T(E_1(\theta)), E_1(\theta))$$

hence

$$\begin{aligned} 0 &= \partial_\theta|_{\theta=\theta_0} g(T(E_1(\theta)), E_2(\theta)) \\ &= g(dT(e_2), e_1) + g(T(e_1), e_2) \\ &= g(T(e_2), e_1) + g(T(e_1), e_2). \end{aligned}$$

Eigenvalues and Eigenvectors

Proof.

We just obtained that

$$g(T(e_1), e_2) = -g(e_1, T(e_2)).$$

But T is self-adjoint and hence

$$g(T(e_1), e_2) = g(e_1, T(e_2))$$

Thus

$$g(T(e_1), e_2) = -g(T(e_1), e_2) \Rightarrow g(T(e_1), e_2) = 0.$$

Therefore $T(e_1) \perp e_2$ and hence

$$T(e_1) = ce_1$$

for some c (possibly $c = 0$ but that's okay).

Eigenvalues and Eigenvectors

Proof.

Finally,

$$c = cg(e_1, e_1) = g(ce_1, e_1) = g(T(e_1), e_1) = \lambda_1.$$

so that

$$T(e_1) = \lambda_1 e_1$$

as claimed. A similar argument gives $T(e_2) = \lambda_2 e_2$ for some λ_2. In fact $\lambda_2 = \max\{g(T(X), X) : X \in \mathbb{S}^1\}$ because:

$$\begin{aligned} g(T(X), X) &= g(T(X^1 e_1 + X^2 e_2), X^1 e_1 + X^2 e_2) \\ &= g(X^1 \lambda_1 e_1 + X^2 \lambda_2 e_2, X^1 e_1 + X^2 e_2) \\ &\underset{e_1, e_2 \text{ o/n}}{=} \lambda_1 X_1^2 + \lambda_2 X^2 \\ &\underset{\lambda_1 = \min}{\leq} \lambda_2 (X_1^2 + X_2^2) \underset{g(X,X)=1}{=} \lambda_2. \end{aligned}$$

Remarks on Eigenvalues and Eigenvectors

- By an induction argument, and using the same ideas, one can prove the general case of n dimensions.
- With respect to the basis of eigenvectors e_1, e_2, T is diagonal:

$$T(X^1 e_1 + X^2 e_2) = X^1 T(e_1) + X^2 T(e_2) = X^1 \lambda_1 e_1 + X^2 \lambda_2 e_2.$$

- As a matrix

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} X^1 \\ X^2 \end{pmatrix} = \begin{pmatrix} \lambda_1 X^1 \\ \lambda_2 X^2 \end{pmatrix}.$$

Quadratic Forms

Let B be a symmetric bi-linear form.
Define the *quadratic form* Q:

$$Q(X) = B(X, X).$$

Q is quadratic in the sense that

$$Q(cX) = B(cX, cX) = c^2 B(X, X) = c^2 Q(X).$$

Notice that

$$\begin{aligned} Q(X+Y) &= B(X+Y, X+Y) = B(X, X+Y) + B(Y, X+Y) \\ &= B(X,X) + B(X,Y) + B(Y,X) + B(Y,Y) \\ &= Q(X) + 2B(X,Y) + Q(Y) \end{aligned}$$

Thus, by symmetry and bi-linearity we may recover B from Q:

$$B(X, Y) = \frac{1}{2}\left[Q(X+Y) - Q(X) - Q(Y)\right].$$